Posets and homotopy

Summer School on General Algebra and Ordered Sets 2009

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The order complex

With a poset P we associate a simplicial complex $\mathcal{K}(P)$ called the *order complex of* P:

vertices – elements of P
simplices – finite chains in P.

$\mathcal{K} : \mathbf{Poset} \to \mathbf{SimpComp}$ is a functor. The homotopy type of the geometric realization of $\mathcal{K}(P)$ is often called the *homotopy type of* P.

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And it has numerous applications:

- Quillen's „Homotopy properties of the poset of nontrivial p-subgroups of a group",
- Lefschetz fixed point theorem for posets,
- $\ldots$

Alexandroff spaces

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Order-preserving maps are continuous maps. Functors $\mathcal{X}, \mathcal{P}$ are mutually inverse.

McCord's results

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McCord (1966) shows that $\mathcal{K}(P)$ and P have the same <u>weak</u> homotopy type. More specifically, there is a weak homotopy equivalence $f : |\mathcal{K}(P)| \to P$.

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For a simplicial complex K, by $\mathcal{O}(K)$ denote its *face poset* (= poset of simplices of K ordered by inclusion).

$P' = \mathcal{O}\mathcal{K}(P)$ is the *barycentric subdivision* of a poset P. P and P' have the same weak homotopy type, but usually different homotopy types.

Homotopy types of finite posets

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Removing beat points of P, one by one, we arrive at a space P^C that has no beat points. It is called a *core* of P. Stong showed that two finite posets are homotopy equivalent if and only if their cores are homeomorphic. In particular, P is contractible iff P^C is a point.

Irreducible points in order theory

Beat points are also known in order theory, under the name of *irreducible points*. In fact, the notion of core was re-discovered by Duffus and Rival (1981). The process of removing irreducible points is called *dismantling*.

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The notion has several applications:

- $P \in \mathrm{FPP}$ iff $P^C \in \mathrm{FPP}$,
- irreducible points' special role in lattices,
- dismantlability equivalent to connectedness (and FPP) of $\mathrm{End}(P), \mathrm{Hom}(P, Q)$,
- $\ldots$

One-point reductions

In analogy to irreducible points, several other notions were introduced. (Osaki 1999, Barmak and Minian 2007-2009)
Let $x \in P$ and

$$\hat{C}_x = \{y \in P : y < x \text{ or } x < y\}.$$

Then x is an:

- *weak point* iff $\hat{C}_x$ is contractible,
- γ*-point* iff $\hat{C}_x$ is homotopically trivial,
- *a-point* iff $\hat{C}_x$ is acyclic.

One-point reductions

x	P and $P \setminus \{x\}$
irreducible	homotopy equivalent
weak point	simple hom. equivalent order complexes
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Irreducible points are well studied in order-theory and dismantlable posets have been shown to have nice properties. What about other one-point reductions?

Homotopy types of infinite posets

Most of the above was developed for finite posets. In the infinite world things complicate.

The relation between dismantlings and homotopy type discovered by Stong rests upon the following observation:

Theorem: If P, Q are finite, then order-preserving maps $f, g : P \to Q$ are homotopic iff there exists a finite sequence $f = f_0, f_1, \dots f_n = g$ with $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$ for all $i = 0, \dots, n-1$.

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If P is a core, then there is no map $f : P \to P$ other than Id_P that is comparable to Id_P. Consequently, two finite cores are homotopy equivalent iff they are isomorphic.

Homotopy types of infinite posets

For infinite P, Q there is in general no known simple way to describe homotopy classes of maps $P \to Q$. One result in this direction is the following (K. 2009):

Theorem: Let $\{f_\alpha : P \to Q\}_{\alpha \leq \gamma}$, where γ is a countable ordinal, be a family of continuous maps such that:

1. if $\alpha < \gamma$, then $f_{\alpha+1}(p)$ is comparable to $f_\alpha(p)$ for every $p \in P$;
2. if α is a limit ordinal, then for every $p \in P$ there exists a $\beta_p^\alpha < \alpha$ such that $f_\beta(p) \leq f_\alpha(p)$ for all $\beta_p^\alpha \leq \beta \leq \alpha$.

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Then f_0 is homotopic to f_γ.

Example: two-way infinite fence is a core, but it is contractible.

Homotopy types of infinite posets

This allowed to extend the homotopy type classification of Stong to the class of countable posets with the property that every sequence (x_n) of distinct elements such that x_i is comparable to x_{i+1} has finite length (*finite-paths spaces*). If the length is bounded from the above by some finite N, then we may omit countablility.

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Another extesion, to chain-complete posets without infinite antichains, is a consequence of the Li-Milner theorem. For other posets – little is known.

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Some directions that may be interesting to explore:

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- order-theoretic properties of one-point reductions of Barmak and Minian,
- generalization to infinite posets (with applications),
- application of known order-theoretic results to the work of Barmak and Minian.

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