Algebras

Algebra = a set with operations:

$$\mathbf{A} = (A, \{f_i \mid i \in I\})$$

n-ary operation on the set A: function $A^n \to A$;

Examples: groups, rings, vector spaces, Boolean algebras, lattices...

Congruences

Congruence on algebra $\mathbf{A}$ is an equivalence relation θ on the set A, preserved by all basic operations of $\mathbf{A}$, i.e.

$$(a_1, b_1) \in \theta,\ (a_2, b_2) \in \theta, \ldots, (a_n, b_n) \in \theta$$

implies

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n) \in \theta))$$

(for f n-ary).
Every algebra with more than 1 element has at least two congruences.

Importance of congruences

Congruences enable *quotients*:
On the set of θ-classes we define the operations be means of representatives:

$$f(a_1/\theta, \ldots, a_n/\theta) = f(a_1, \ldots, a_n)/\theta.$$

This gives rise to a new algebra of the same type as $\mathbf{A}$, which is a simplified image of the algebra A.

For instance, $\mathbb{Z}/(\operatorname{mod} n) = \mathbb{Z}_n$.

Example: groups

$\mathbf{A}$ ... a group;
Every normal subgroup $\mathbf{B}$ of $\mathbf{A}$ determines a congruence

$$\theta = \{(a,b) \in A^2 \mid ab^{-1} \in B\}.$$

(That's why we speak about a factorization of a group by a normal subgroup.)

Similarly: rings, vector spaces

Example: Boolean algebras

$\mathbf{B} = (B; \cup, \cap, ', 0, 1)$
$(B \subseteq \mathcal{P}(X))$;
Ideal: a subset $I \subseteq B$ such that

- if $M \in I$, $N \subseteq M$, then $N \in I$;
- if $M, N \in I$, then $M \cup N \in I$.

Every ideal determines a congruence (and vice versa):

$$\theta = \{(M, N) \in B^2 \mid (M \cap N') \cup (M' \cap N) \in I\}.$$

Example: chains

Consider $(\mathbb{Z}, \max, \min)$ (a distributive lattice)

Fact: Congruences are equivalences, whose all classes are intervals.

Congruence lattices

Congruences on an algebra $\mathbf{A}$ can be ordered by the "refinement" relation (= set inclusion):

$$\varphi \leq \theta \text{ ak } (x\varphi y \text{ implies } x\theta y).$$

We obtain a complete lattice $\mathrm{Con}\,\mathbf{A}$.

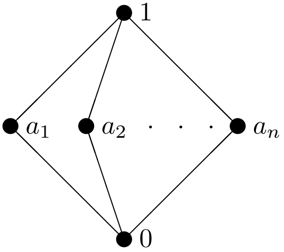
Congruence lattices

For $(\mathbb{Z}, +, \cdot)$:

$$(\mathrm{mod}\, n) \leq (\mathrm{mod}\, m) \quad \text{if } m|n.$$

So: $\mathrm{Con}\,\mathbb{Z}$ is (isomorphic to) the set of all nonnegative integers, the smallest element is $(\mathrm{mod}\, 0)$, the largest $(\mathrm{mod}\, 1)$, the infimum is the LCM and the supremum is the GCD.

Let $\mathbf{A}$ be the 2-dimensional vector space over a field F. Every nontrivial congruence looks the same: its congruence classes are mutually parallel lines. So $\mathrm{Con}\,\mathbf{A}$ looks as follows:



The number of elements in the middle layer is equal to the number of the lines containing 0. For a finite F it is $n = |F| + 1$.

Congruence lattices

Is every lattice isomorphic to the congruence lattice of some algebra?

Theorem

(G. Grätzer, E. T. Schmidt) *A lattice is isomorphic to the congruence lattice of some algebra if and only if it is algebraic.*

What about congruence lattices of special kinds of algebras?

Congruence lattices

Open problem: Is every *finite* lattice (isomorphic to) the congruence lattice of some *finite* algebra?

Equivalent group formulation: Is every *finite* lattice (isomorphic to) an interval in the subgroup lattice of a *finite* group?

Congruence lattices

One solved problem: Is every *distributive* algebraic lattice (isomorphic to) the congruence lattice of some lattice?

Partial positive results (R. P. Dilworth, E. T. Schmidt, A. Huhn...), but
Final answer: no (F. Wehrung 2005)

General problem

Problem. For a given class $\mathcal{K}$ of algebras describe Con $\mathcal{K}$ =all algebras isomorphic to Con A for some $A \in \mathcal{K}$.

Or, at least,

for given classes $\mathcal{K}$, $\mathcal{L}$ determine if Con $\mathcal{K} =$ Con $\mathcal{L}$ (Con $\mathcal{K} \subseteq$ Con $\mathcal{L}$)

and, if Con $\mathcal{K} \nsubseteq$ Con $\mathcal{L}$, determine

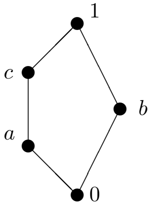
$$\mathrm{Crit}(\mathcal{K}, \mathcal{L}) = \min\{\mathrm{card}(L_c) \mid L \in \mathrm{Con}\,\mathcal{K} \setminus \mathrm{Con}\,\mathcal{L}\}$$

($L_c =$ compact elements of L)

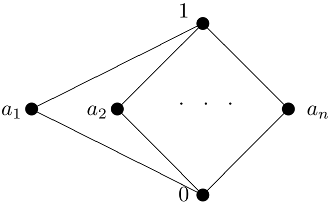
Some critical points

We are especially interested in the case when $\mathcal{K}$ and $\mathcal{L}$ are congruence-distributive varieties (in most results also finitely generated). For instance,
$\mathrm{Crit}(\mathbf{N}_5, \mathbf{M}_3) = 5$,
$\mathrm{Crit}(\mathbf{M}_3, \mathbf{N}_5) = \mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) = \aleph_0$,
$\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$,
$\mathrm{Crit}(\mathbf{Maj}, \mathbf{Lat}) = \aleph_2$.
($\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{M}_4$, $\mathbf{D}$ are well-known lattice varieties, $\mathbf{Lat}$ = all lattices, $\mathbf{Maj}$ = all majority algebras.)
P. Gillibert: under some reasonable finiteness conditions, the critical point between two varieties cannot be larger than $\aleph_2$.

N_5 and M_n



N_5



M_n

Topological approach

$\mathrm{M}(L)$....completely meet-irreducible elements of a lattice L
($a = \inf X$ implies $a \in X$)
Fact: if L is algebraic, then every element is a meet of completely meet-irreducible elements.
Topology on $\mathrm{M}(L)$: all sets of the form

$$\mathrm{M}(L) \cap \uparrow x = \{a \in \mathrm{M}(L) \mid a \geq x\}$$

are closed.

Theorem

If L is distributive algebraic, then $L \cong \mathcal{O}(\mathrm{M}(L))$. (The lattice of all open subsets of $\mathrm{M}(L)$.

Topological approach

Sometimes the properties of $\operatorname{Con} A$ are more effectively expressed as topological properties of $\mathrm{M}(\operatorname{Con} A)$. A sample:

- If $A \in \mathbf{D}$ then $\mathrm{M}(\operatorname{Con} A)$ is Hausdorff.
- There exists a countable $B \in \mathbf{M}_3$ such that $\mathrm{M}(\operatorname{Con} B)$ is not Hausdorff.
- Therefore, $\operatorname{Crit}(\mathbf{M}_3, \mathbf{D}) \leq \aleph_0$.

The topological approach was used to establish e.g. $\operatorname{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$. (But the argument is much more complicated.)

Con functor

The Con functor:

For any homomorphism of algebras $f: A \to B$ we define

$$\mathrm{Con}\, f: \mathrm{Con}\, A \to \mathrm{Con}\, B$$

by
$\alpha \mapsto$ congruence generated by $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Fact. $\mathrm{Con}\, f$ preserves $\vee$ and 0, not necessarily $\wedge$.

Lifting of semilattice morphisms

Let

- $\varphi : \ S \to T$ be a $(\vee, 0)$-homomorphisms of lattices;
- $f : \ A \to B$ be a homomorphisms of algebras.

We say that f *lifts* φ, if there are isomorphisms $\psi_1 : \ S \to \operatorname{Con} A$, $\psi_2 : \ T \to \operatorname{Con} B$ such that

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ \psi_1 \Big\downarrow & & \psi_2 \Big\downarrow \\ \operatorname{Con} A & \xrightarrow{\operatorname{Con} f} & \operatorname{Con} B \end{array}$$

commutes.

A generalization: lifting of semilattice diagrams

Results of P. Gillibert 1

Let $\mathcal{K}$, $\mathcal{L}$ be finitely generated congruence distributive varieties.

Theorem

TFAE

- $\operatorname{Con}\mathcal{K} \not\subseteq \operatorname{Con}\mathcal{L}$;
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by* $\{0,1\}^n$ *(for some* n*) liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

Results of P.Gillibert 2

Theorem

(2) implies (1), where

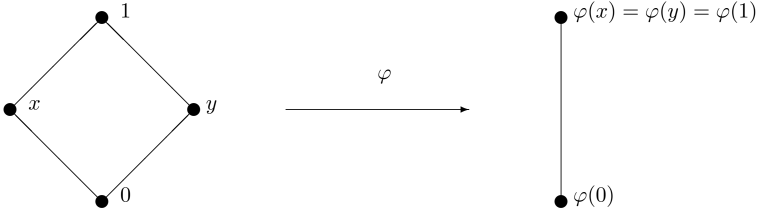
- $\mathrm{Crit}(\mathcal{K}, \mathcal{L}) \leq \aleph_n$*;*
- *there exists a diagram of finite* $(\vee, 0)$*-semilattices indexed by a product of* $n+1$ *finite chains liftable in* $\mathcal{K}$ *but not in* $\mathcal{L}$

If $n = 0$ *then also (1)*$\Longrightarrow$ *(2).*

Question. What about (1)$\Longrightarrow$(2) for $n > 0$?

Example

The semilattice homomorphism



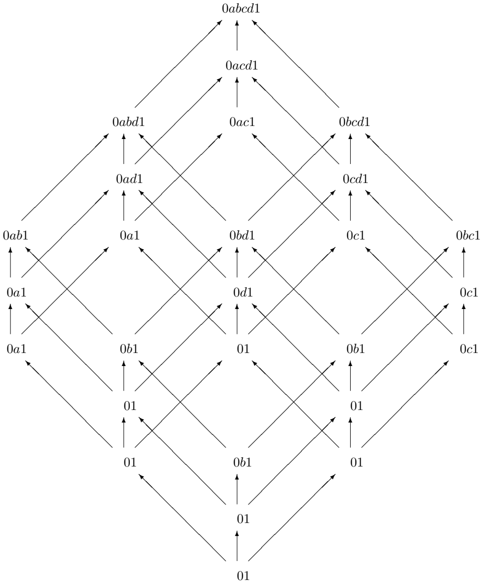
has a lifting in $\mathbf{M}_3$ (the embedding of a 3-element chain into M_3 lifts it), but not in $\mathbf{D}$. Therefore, $\mathrm{Crit}(\mathbf{M}_3, \mathbf{D}) \leq \aleph_0$.

Critical point aleph2

We know that $\mathrm{Crit}(\mathbf{M}_4, \mathbf{M}_3) = \aleph_2$. Is there a diagram indexed by a product of 3 finite chains liftable in $\mathbf{M}_4$ but not in $\mathbf{M}_3$?

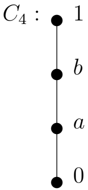
Yes, it is on the next slide.

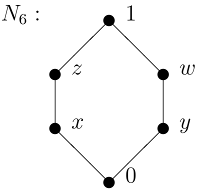
M3 versus M4


0abcd1
0acd1
0abd1
0ac1
0bcd1
0ad1
0cd1
0ab1
0a1
0bd1
0c1
0bc1
0a1
0d1
0c1
0a1
0b1
01
0b1
0c1
01
01
01
0b1
01
01
01

Critical point $\aleph_1$

Let $\mathbf{C}_4^*$ and $\mathbf{N}_6^*$ be the varieties generated by the bounded lattices C_4 and N_6 with an additional unary operation:
on C_4 ... $f(0) = 0$, $f(a) = b$, $f(b) = a$, $f(1) = 0$;
on N_6 ... $180°$ rotation $(f(x) = w...)$.





Critical points $\aleph_1$

Theorem

(1) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{N}_5) = \aleph_1$;

(2) $\mathrm{Crit}(\mathbf{N}_5, \mathbf{N}_6^*) = \aleph_0$.

(3) $\mathrm{Crit}(\mathbf{N}_6^*, \mathbf{C}_4^*) = \aleph_1$;

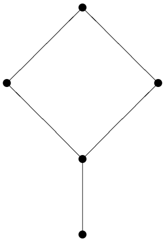
(4) $\mathrm{Crit}(\mathbf{C}_4^*, \mathbf{N}_6^*) = \infty$.

Question

What is the mechanism behind these examples?

N6 versus N5

Both N_5 and N_6^* have the same congruence lattice, but N_6^* has an automorphism h (the vertical symmetry), such that $\mathrm{Con}_c\, h$ interchanges α and β:



N6 versus N5

Below: $\mathcal{D}$ is the diagram in $\mathbf{N}_6^*$, so that $\operatorname{Con}\mathcal{D}$ has a lifting in $\mathbf{N}_6^*$ but - no lifting in $\mathbf{N}_5$.

$$
\begin{tikzcd}
& N_6^* \times N_6^* & \\
N_6^* \arrow[ur, "\mathrm{id}\times\mathrm{id}"] & & N_6^* \arrow[ul, "\mathrm{id}\times h"'] \\
& \{0,1\} \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] &
\end{tikzcd}
$$

Observation

Every automorphism $f: A \to A$ induces an automorphism $\operatorname{Con}_c f: \operatorname{Con}_c A \to \operatorname{Con}_c A$. These induced automorphisms form a subgroup of the automorphism group of $\operatorname{Con}_c A$. And this subgroup has an influence on the class $\operatorname{Con} \mathbf{A}$, where A is the variety generated by A.