(?) On the endomorphism kernel property

Emília Halušková

seminár Matematického ústavu SAV, Košice

16. 3. 2017

Outline

J. Chvalina, O. Kopeček, M. Novotný:
Homomorphic transformations - why and possible ways to how,
Brno, 2012.

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

$h(x) := 2x + 1$

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

$h(x) := 2x + 1$

Find all homomorphisms from $(\mathbb{Z}, h)$ into $(\mathbb{Z}, h)$.

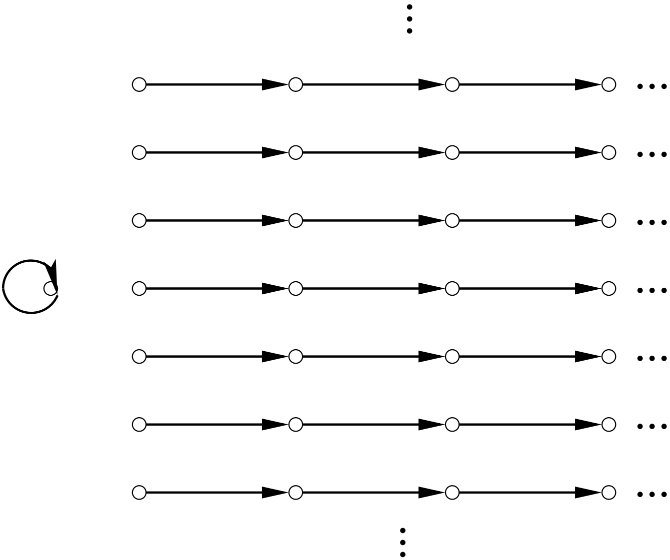
Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

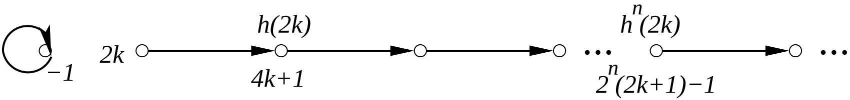
$$2f(x) - f(2x+1) + 1 = 0.$$

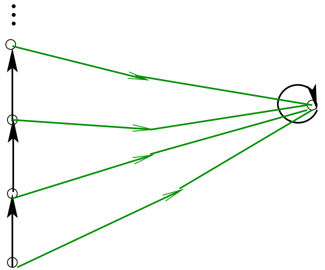
$h(x) := 2x + 1$

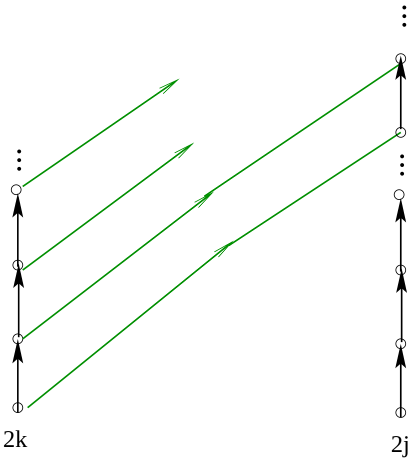
Find all homomorphisms from $(\mathbb{Z}, h)$ into $(\mathbb{Z}, h)$.

Find all endomorphisms of $(\mathbb{Z}, h)$.




−1
2k
h(2k)
4k+1
$h^n(2k)$
$2^n(2k+1)-1$
...




2k
2j

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Solution:

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Solution:

$h(x) := 2x + 1$
Choose $f(2k)$ arbitrarily for every $k \in \mathbb{Z}$;
$f(2k+1)$ are uniquely determined by the algebra $(\mathbb{Z}, h)$.

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(x+1) + 1 = 0.$$

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(x+1) + 1 = 0.$$

$h(x) := 2x + 1$
$g(x) := x + 1$

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(x+1) + 1 = 0.$$

$h(x) := 2x + 1$
$g(x) := x + 1$

Find all homomorphisms from $(\mathbb{Z}, g)$ into $(\mathbb{Z}, h)$.

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(x+1) + 1 = 0.$$

$h(x) := 2x + 1$
$g(x) := x + 1$

Find all homomorphisms from $(\mathbb{Z}, g)$ into $(\mathbb{Z}, h)$.

Solution:

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(x+1) + 1 = 0.$$

$h(x) := 2x + 1$
$g(x) := x + 1$

Find all homomorphisms from $(\mathbb{Z}, g)$ into $(\mathbb{Z}, h)$.

Solution:
– exactly one: $f(x) = -1$

Find all linear functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Find all linear functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Solution:

$$f(x) = ax + a - 1, \; a \in \mathbb{Z}$$

Find all linear functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Solution:

$$f(x) = ax + a - 1, \; a \in \mathbb{Z}$$

How does f work on $(\mathbb{Z}, h)$?

Find all linear functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$2f(x) - f(2x+1) + 1 = 0.$$

Solution:

$$f(x) = ax + a - 1,\ \ a \in \mathbb{Z}$$

How does f work on $(\mathbb{Z}, h)$?

How does the algebra $(f(\mathbb{Z}), h)$ look like?

How does the algebra $(f(\mathbb{Z}), h)$ look like?

$$f(x) = ax + a - 1, \; a \in \mathbb{Z}$$

How does the algebra $(f(\mathbb{Z}), h)$ look like?

$$f(x) = ax + a - 1,\ \ a \in \mathbb{Z}$$

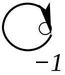
Let $a = 0$.

How does the algebra $(f(\mathbb{Z}), h)$ look like?

$$f(x) = ax + a - 1, \; a \in \mathbb{Z}$$

Let $a = 0$.

$(f(\mathbb{Z}), h)$



How does the algebra $(f(\mathbb{Z}), h)$ look like?

$$f(x) = ax + a - 1, \;\; a \in \mathbb{Z}$$

Let $a \neq 0$.

How does the algebra $(f(\mathbb{Z}), h)$ look like?

$$f(x) = ax + a - 1, \ a \in \mathbb{Z}$$

Let $a \neq 0$.
Take $n \in \mathbb{N}, k \in \mathbb{Z}$ such that

$$a = 2^n(2k + 1)$$

How does the algebra $(f(\mathbb{Z}), h)$ look like?

$$f(x) = ax + a - 1,\ \ a \in \mathbb{Z}$$

Let $a \neq 0$.
Take $n \in \mathbb{N}, k \in \mathbb{Z}$ such that

$$a = 2^n(2k+1)$$

Then $f(0) = a - 1 = h^n(2k)$,
$f(2) = 3a - 1 = h^n(2(3k+1))$.

How does the algebra $(f(\mathbb{Z}), h)$ look like?

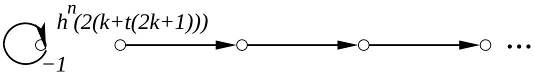
$$f(x) = ax + a - 1, \;\; a \in \mathbb{Z}$$

Let $a \neq 0$.
Take $n \in \mathbb{N}, k \in \mathbb{Z}$ such that

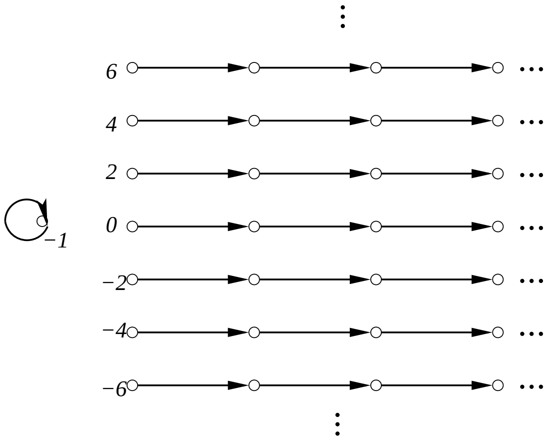
$$a = 2^n(2k+1)$$

Then $f(0) = a - 1 = h^n(2k)$,
$f(2) = 3a - 1 = h^n(2(3k+1))$.

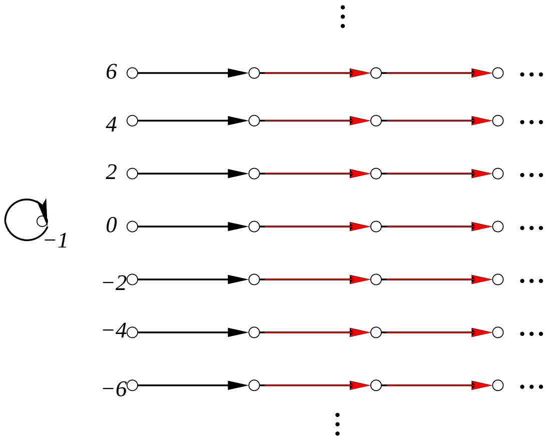
Further, $(f(\mathbb{Z}), h)$



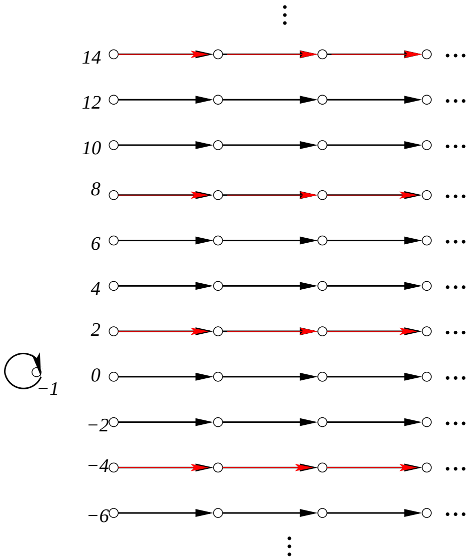
$a = \pm 1$



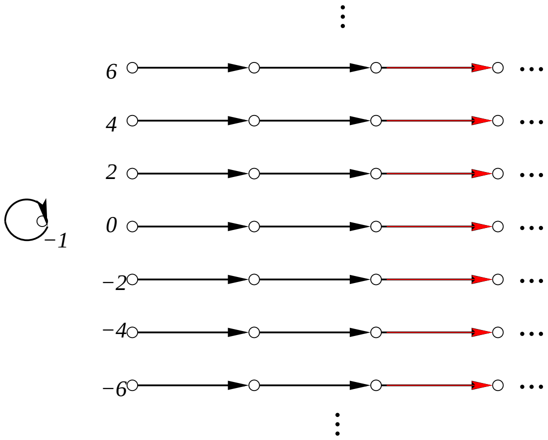
$a = \pm 2$



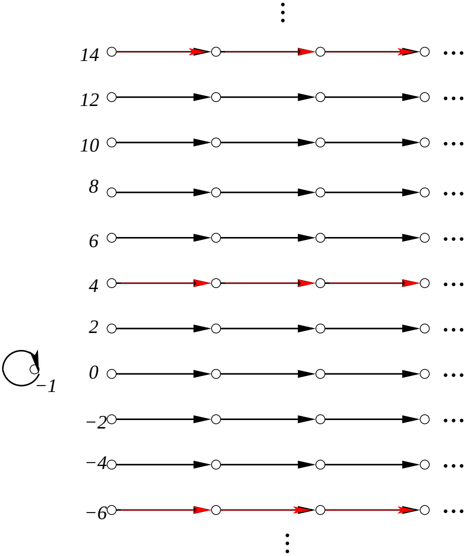
$a = \begin{smallmatrix}+\\-\end{smallmatrix}\, 3$



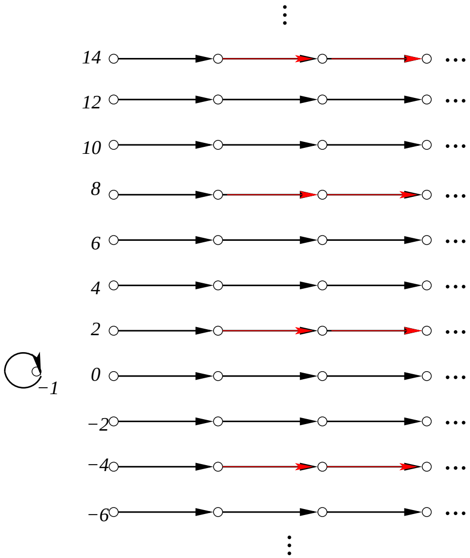
$a = \begin{smallmatrix}+\\-\end{smallmatrix}\, 4$



$a = \pm 5$



$a = \pm 6$



$\mathcal{A}$ ..... algebra
$End\mathcal{A}$ ..... set of all endomorphisms of $\mathcal{A}$
$Con\mathcal{A}$ ..... set of all congruences of $\mathcal{A}$

$\mathcal{A}$ ..... algebra
$End\mathcal{A}$ ..... set of all endomorphisms of $\mathcal{A}$
$Con\mathcal{A}$ ..... set of all congruences of $\mathcal{A}$

We have

$$\{Ker\varphi, \varphi \in End\mathcal{A}\} \subseteq Con\mathcal{A}$$

$\mathcal{A}$ ..... algebra
$End\mathcal{A}$ ..... set of all endomorphisms of $\mathcal{A}$
$Con\mathcal{A}$ ..... set of all congruences of $\mathcal{A}$

$\mathcal{A}$ has EKP if

$$\{Ker\varphi, \varphi \in End\mathcal{A}\} = Con\mathcal{A}$$

$\mathcal{A}$ ..... algebra
$End\mathcal{A}$ ..... set of all endomorphisms of $\mathcal{A}$
$Con\mathcal{A}$ ..... set of all congruences of $\mathcal{A}$

$\mathcal{A}$ has EKP if

$$\{Ker\varphi, \varphi \in End\mathcal{A}\} = Con\mathcal{A}$$

Lemma

$\mathcal{A}$ has EKP iff
every homomorphic image of $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{A}$.

$$\mathcal{A} = (A, f), \ f : A \to A$$

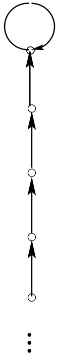
$\mathcal{A} = (A, f), \; f : A \to A$

EXAMPLE:

if $f \in \{id_A, const\}$ then $\mathcal{A}$ has EKP

$\mathcal{A} = (A, f), \; f : A \to A$

EXAMPLE:



Description of all finite

Let A be finite.

Theorem (2015)

$\mathcal{A}$ has EKP iff

if $f \notin \{id_A, const\}$, then $\mathcal{A}$ contains exactly 1 nontrivial component $\mathcal{B} = (B, f)$ and $\mathcal{B}$ satisfies that

a) $f(c) = c$ *for some* $c \in B$,

b) *if* $B - f^{-1}(c) \neq \emptyset$, *then the set* $B - f^{-1}(c)$ *is a chain of* $\mathcal{B}$.

Description for f injective

Let f be injective.

Theorem

$\mathcal{A}$ has EKP iff

a) *Every component of $\mathcal{A}$ has a cycle.*

b) *For arbitrary $k, l, m \in \mathbb{N}$ such that l divides k, $l \neq k$ the following conditions are satisfied:*

  b1) *If $\mathcal{A}$ possesses m cycles of length k, then $\mathcal{A}$ possesses $\aleph_0$ cycles of length l.*

  b2) *If κ is an infinite cardinal number and $\mathcal{A}$ possesses κ cycles of length k, then $\mathcal{A}$ possesses κ cycles of length l.*

Description of some connected

Let $\mathcal{A}$ be connected with a cycle $\{c\}$ and $f^{-1}(c)$ be finite.

Theorem

$\mathcal{A}$ has EKP iff

if $A \setminus f^{-1}(c) \neq \emptyset$, then $A \setminus f^{-1}(c)$ is a chain of $\mathcal{A}$.

Description of some connected

Let $\mathcal{A}$ be countable with a cycle $\{c\}$ and $f^2(a) = c$ for every $a \in A$. Put

$$D = f^{-1}(c) \setminus \{c\},$$

$$R = \{a \in D : f^{-1}(a) \text{ is nonempty and finite }\},$$

$$P = \{a \in D : f^{-1}(a) \text{ is infinite }\}.$$

Assume that D is infinite.

Theorem

$\mathcal{A}$ has EKP iff

if P is finite, then $\|R\| \leq 1$.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.

Lemma

If $\mathcal{A}$ is not connected and $\mathcal{B} = (B, f)$ is a component of $\mathcal{A}$, then the algebra $(A \setminus B, f)$ has EKP.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.

Ord .... the set of all ordinal numbers
∞ .... a symbol such that $\infty > \lambda$ for every $\lambda \in Ord$
$s(a)$ .... source degree of an element $a \in A$

Lemma

Let $\lambda \in Ord \cup \{\infty\}$.
If $B = \{a \in A : s(a) \geq \lambda\}$, then the algebra (B, f) has EKP.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.

Ord .... the set of all ordinal numbers
∞ .... a symbol such that $\infty > \lambda$ for every $\lambda \in Ord$
$s(a)$ .... source degree of an element $a \in A$

Lemma

Let $\lambda \in Ord \cup \{\infty\}$.
If $B = \{a \in A : s(a) \geq \lambda\}$, then the algebra (B, f) has EKP.

Remark: If $\lambda \in \mathbb{N}$, then $B = f^{\lambda}(A)$.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.
Let every component of $\mathcal{A}$ contain a cycle.

Subalgebras with EKP

Suppose that $\mathcal{A} = (A, f)$ has EKP.
Let every component of $\mathcal{A}$ contain a cycle.

$\delta(a)$ .... distance of $a \in A$ from a cycle of $\mathcal{A}$

Lemma

Let $k \in \mathbb{N}_0$*.*
If $B = \{a \in A : \delta(a) \leq k\}$*, then the algebra* (B, f) *has EKP.*

1. Finite monounary algebras with EKP are very simple and they are closed with respect to subalgebras.

1. Finite monounary algebras with EKP are very simple and they are closed with respect to subalgebras.

2. Infinite monounary algebras with EKP are more complicated. Several classes of them are described.

1. Finite monounary algebras with EKP are very simple and they are closed with respect to subalgebras.

2. Infinite monounary algebras with EKP are more complicated. Several classes of them are described.

3. Every monounary algebra with EKP contains "well defined" subalgebras with EKP.

1 T.S. Blyth, J. Fang and H. J. Silva: *EKP in finite distributive lattices and de Morgan algebras*, Communications in Algebra, 32 (6), (2004), 2225-2242.

2 T.S. Blyth, H. J. Silva: *SEKP in Ockham algebras*, Communications in Algebra, 36 (5), (2004), 1682-1694.

3 T.S. Blyth, J. Fang and L.-B. Wang: *SEKP in distributive double p-algebras*, Sci. Math. Jpn. 76 (2), (2013), 227-234.

4 G. Fang and J. Fang: *SEKP in distributive p-algebras*, Southeast Asian Bull. of Math. 37, (2013), 491-497.

5 J. Fang, Z.-J. Sun: *Semilattices with the SEKP*, Algebra Univ. 70 (4), (2013), 393-401.

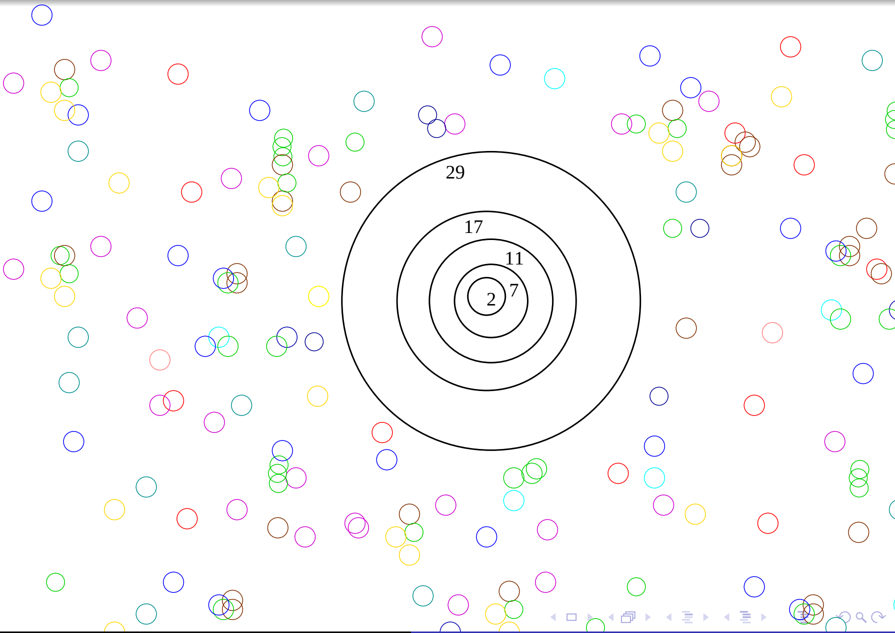
6 H. Gaitán, Y.J. Cortés: *EKP in finite Stone algebras*, JP J. of Algebra, Number Theory and Appl., 14(1), (2009), 51-64.

7 J. Guričan: *EKP for modular p-algebras and Stone lattices of order n*, JP J. of Algebra, Number Theory and Appl., 25(1), (2012), 69-90.

8 J. Guričan: *A note on EKP*, JP J. of Algebra, Number Theory and Appl., 33 (2), (2014), 133-139.

9 J. Guričan: *SEKP for Brouwerian algebras*, JP J. of Algebra, Number Theory and Appl., 36(3), (2015), 241-258.

10 J. Guričan, M. Ploščica: *SEKP for modular p-algebras and for distributive lattices*, Algebra Univ., 75 (2), (2016), 243-255.


29
17
11
7
2