# Causal Dynamics of Discrete Manifolds

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# Motivation

- Discrete/quantized versions of General Relativity
  - discretized as simplicial complexes (Regge-calculus)
  - or in the basis of spin networks graphs (Loop Quantum Gravity).
- Other discrete models in physics
  - lattice-gas models
  - ...

# Graph dynamics

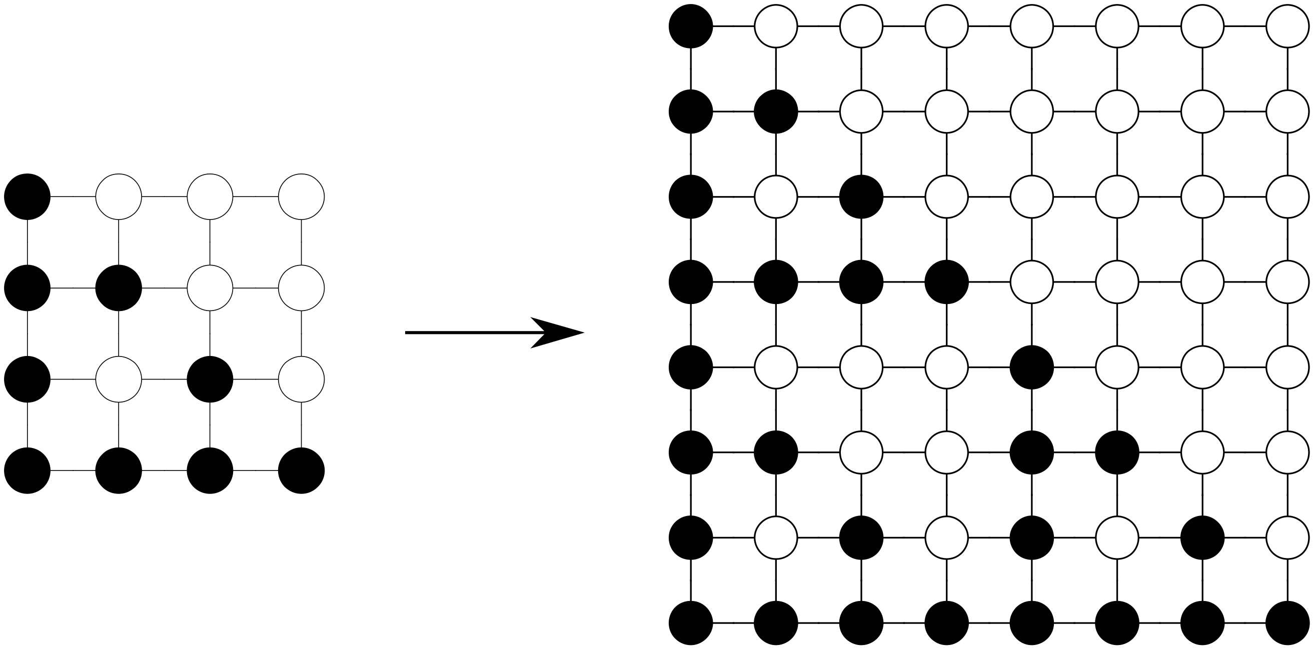
A generalized discrete space with a notion of “proximity”: **graph**

which evolves in time, subject to two natural constraints:

- **causality**: the evolution does not propagate information too fast
- **homogeneity**: it acts everywhere the same (translation invariance)

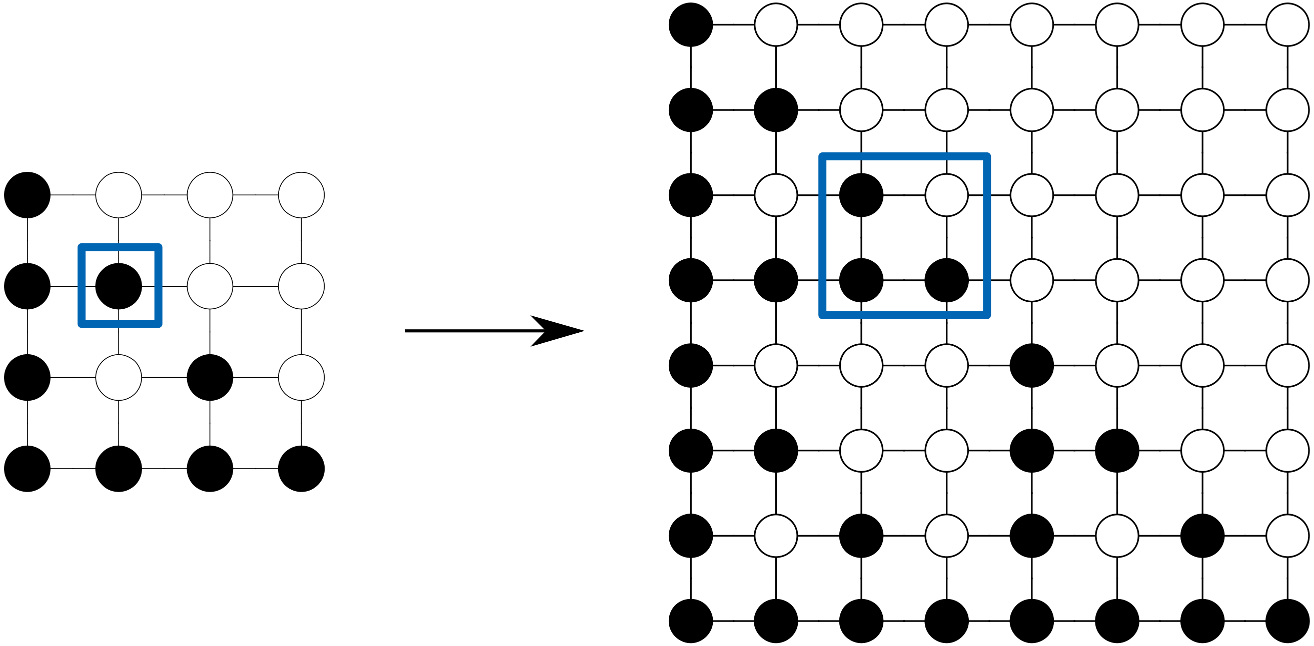
P. Arrighi, G. Dowek. Causal graph dynamics. Information and Computation, 223:78–93, 2013

# Graph dynamics



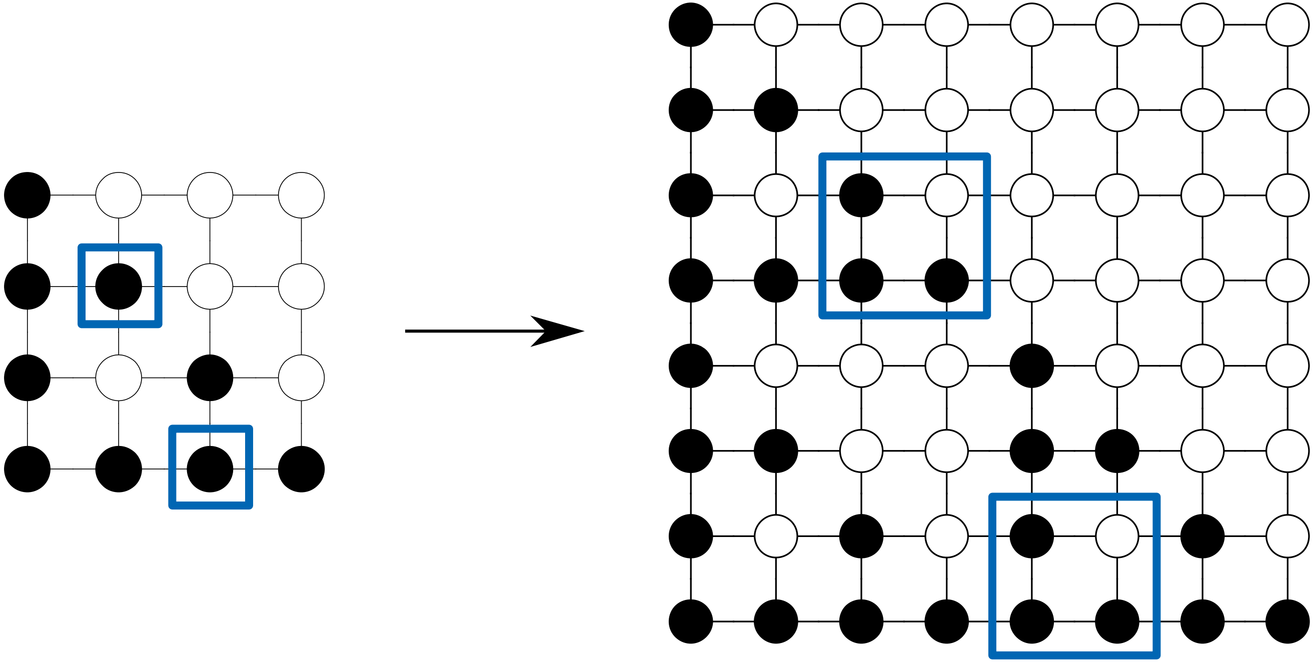
# Graph dynamics

Causality



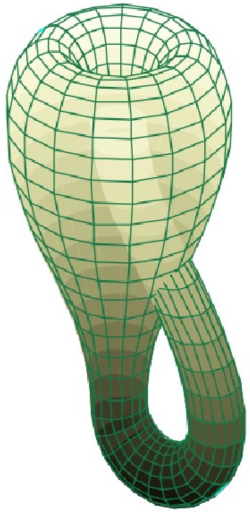
# Graph dynamics

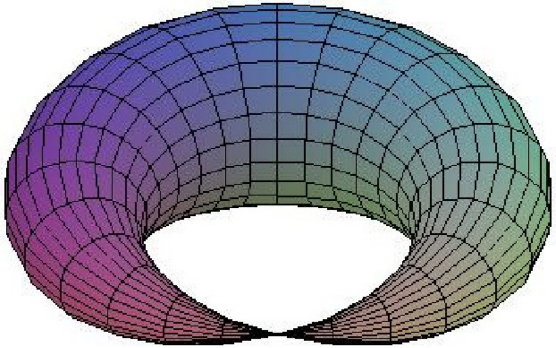
Causality + Homogeneity



# Motivation

Physical models have geometrical content → Simplicial Complexes





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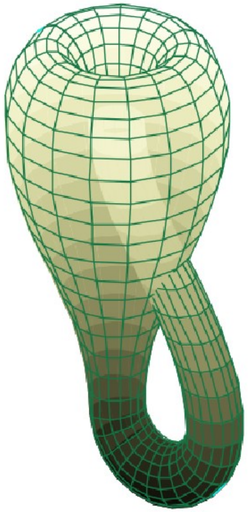
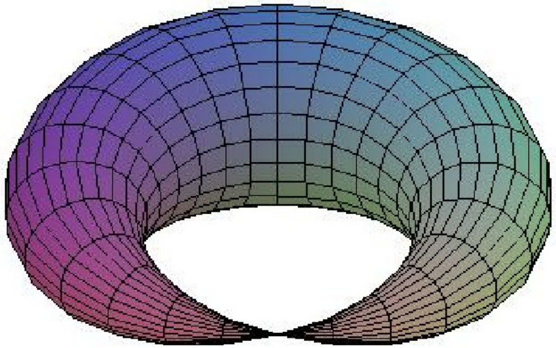
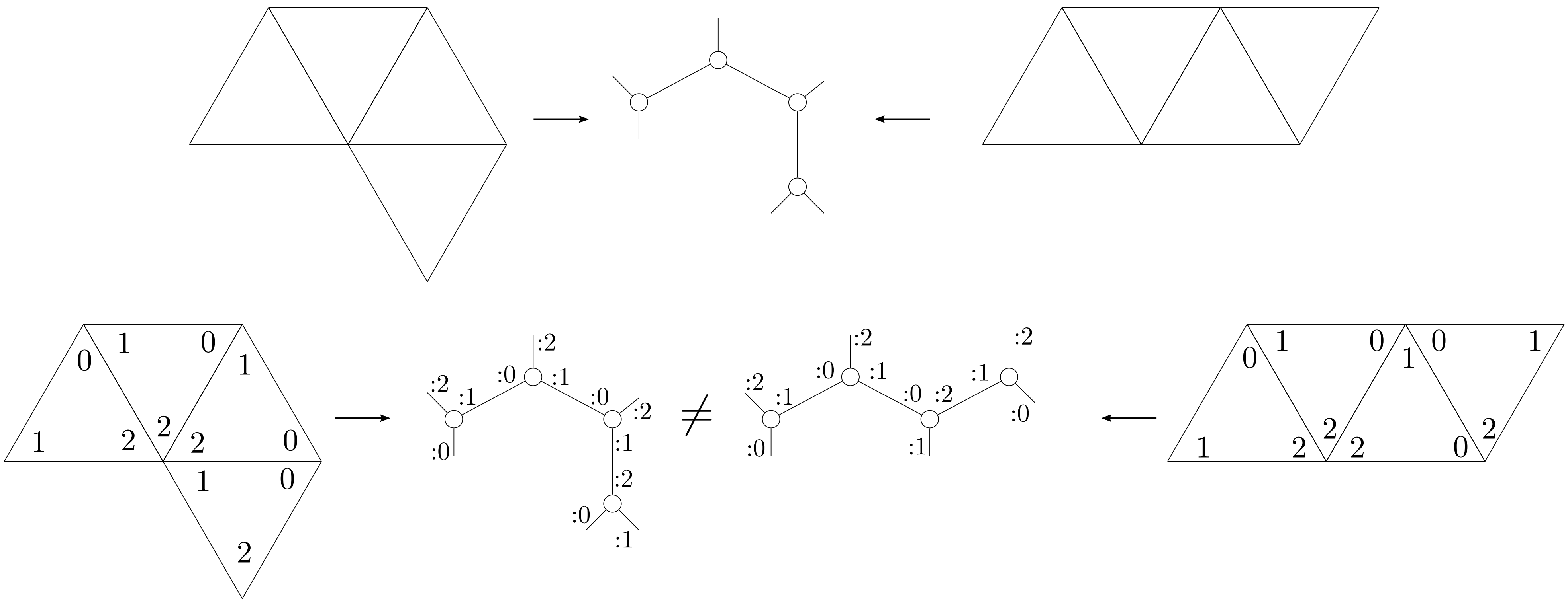


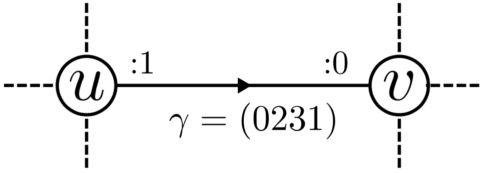


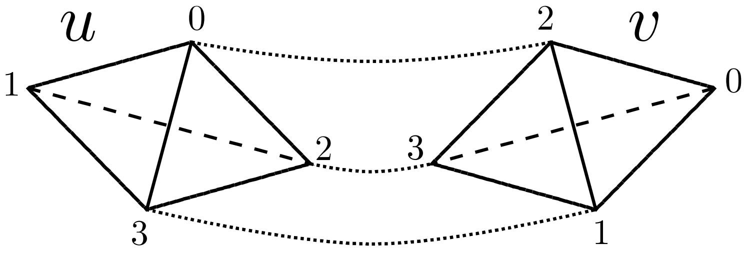
**Encode complexes as graphs!**

# Complexes as graphs



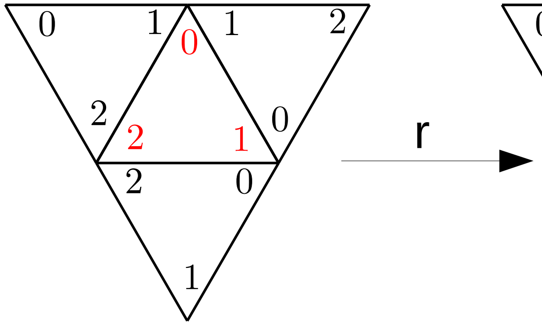
# Complexes as graphs: dimension ≥ 3

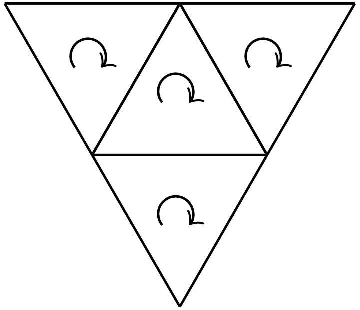



$$u : 0 \equiv v : \gamma(0) = v : 2$$
$$u : 2 \equiv v : \gamma(2) = v : 3$$
$$u : 3 \equiv v : \gamma(3) = v : 1$$

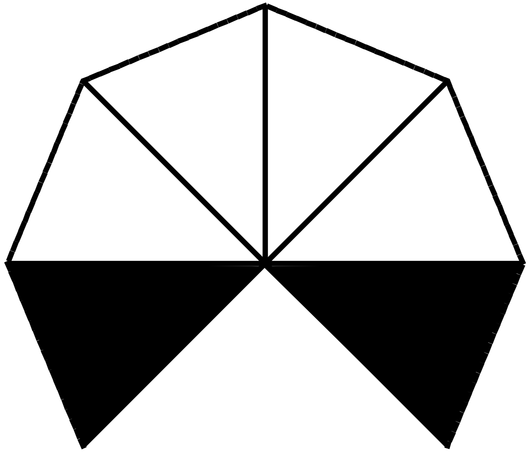
# Complexes as graphs: Rotation Equivalence



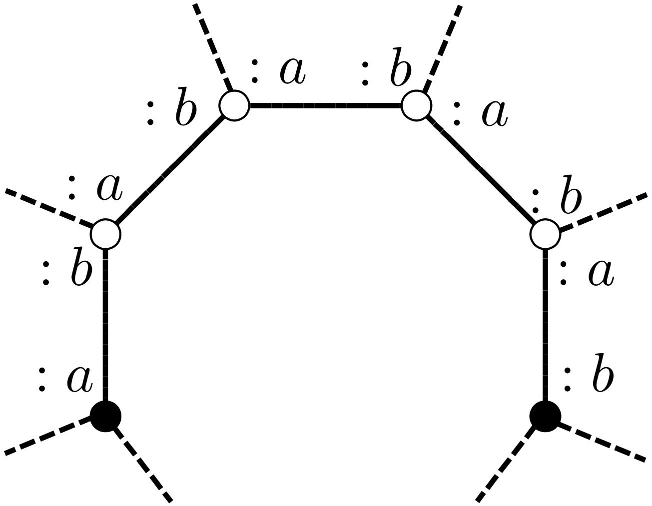



Oriented simplicial complexes
correspond to the equivalence classes

# Graph distance vs Geometrical distance

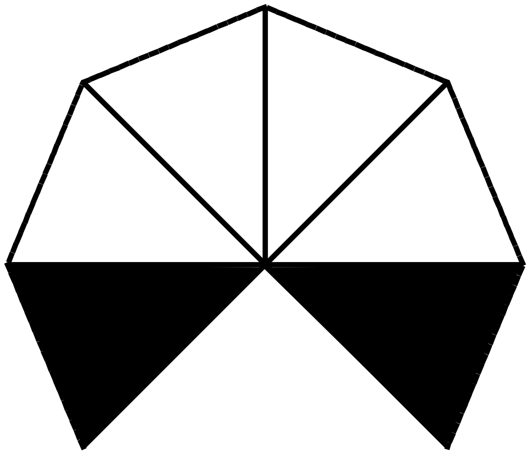


geometrical distance = 1




graph distance = 5

# Graph distance vs Geometrical distance



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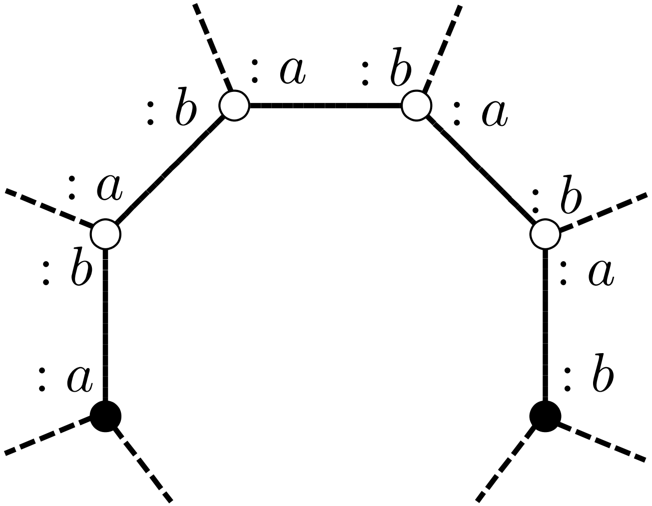



graph distance = 5

From the **dynamical** point of view:

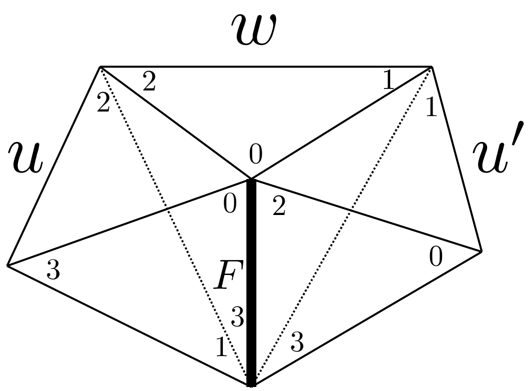
- ignore geometrical distance → Causal Dynamics Complexes (CDC)
- take geom. dist. into account → CDC + further restrictions

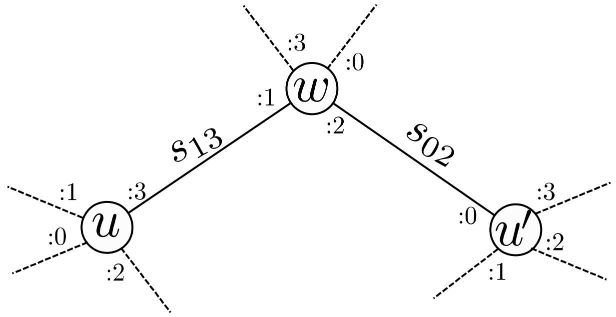
# Graph distance vs Geometrical distance

**Definition (equivalent faces)**
Two k-faces F at vertex u and F’ at vertex u’ are said to be equivalent if and only if they are related by a hinge, i.e. if and only if there exists is a path $(u_i :p_i, \gamma_{i+1}, u_{i+1}:q_{i+1}) \in E(G)$ with $i = 0\ldots m$, $u_0 = u$, $u_{m+1} = u'$, such that:

$$p_i, q_i \notin (\textstyle\prod^i_{j=1}\gamma_j)(F) \qquad \text{and} \qquad F' = (\textstyle\prod^m_{j=1}\gamma_j)(F)$$

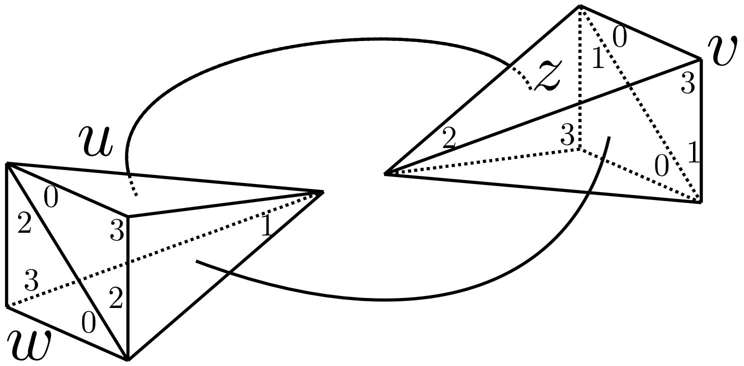
where $p_i = p_0 , \ldots , p_m$ , whereas $q_i = q_1 , \ldots , q_{m+1}$.

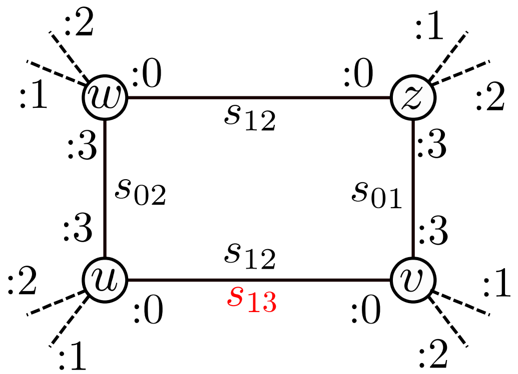



$$F = \{u : 0, u : 1\} \equiv F' = \{u' : 2, u' : 3\}$$

# Torsion



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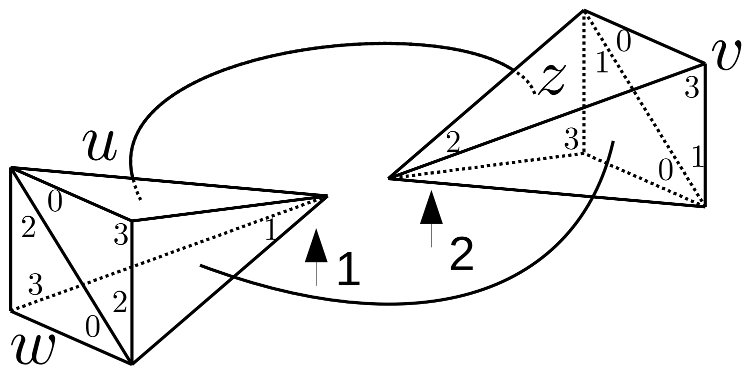


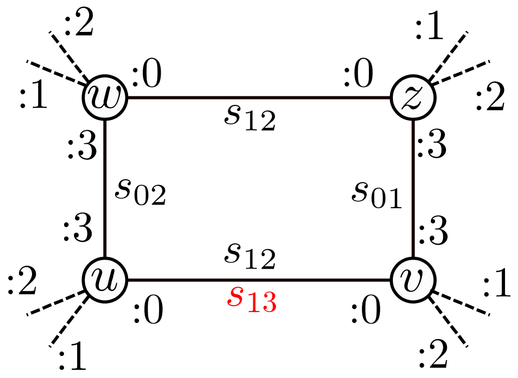

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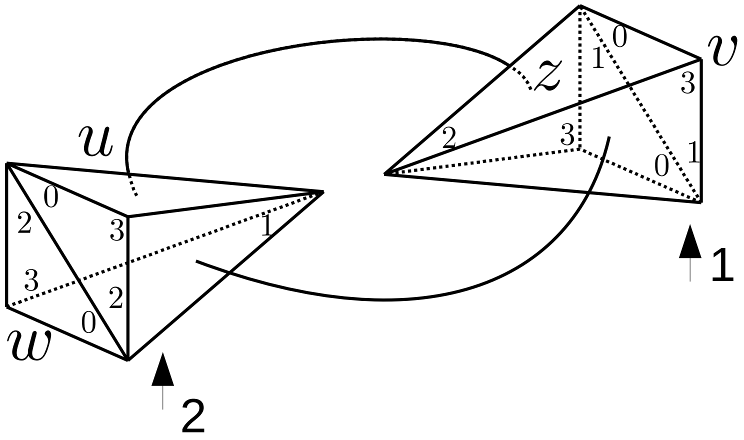


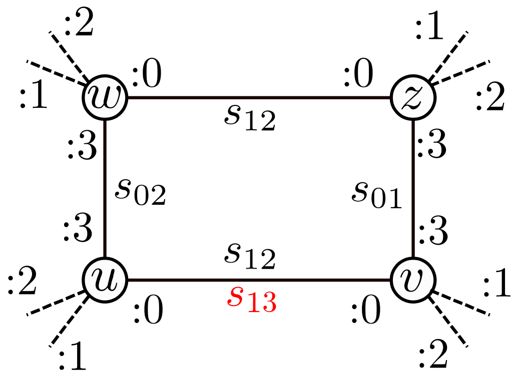

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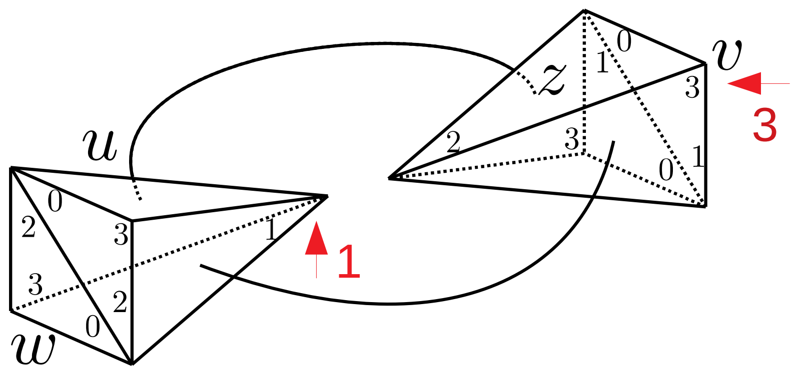


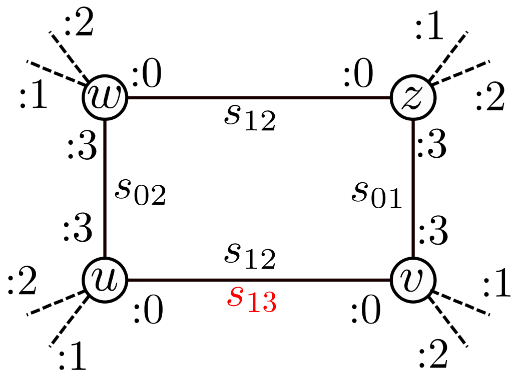

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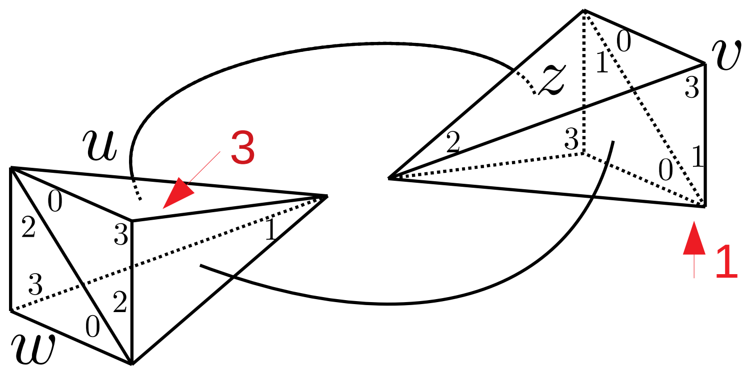


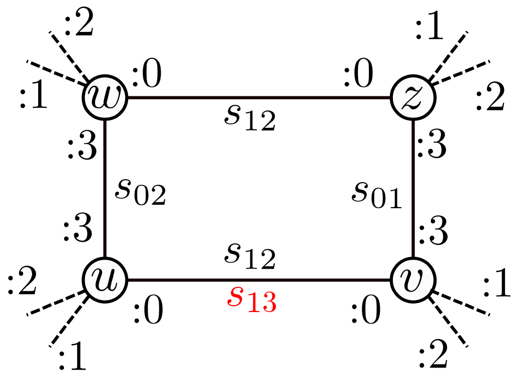

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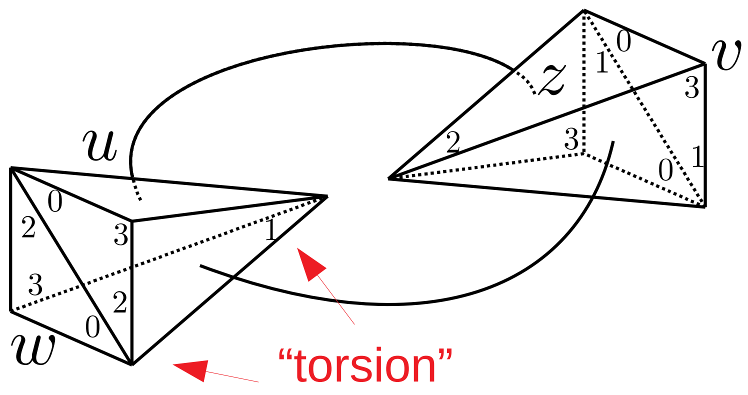


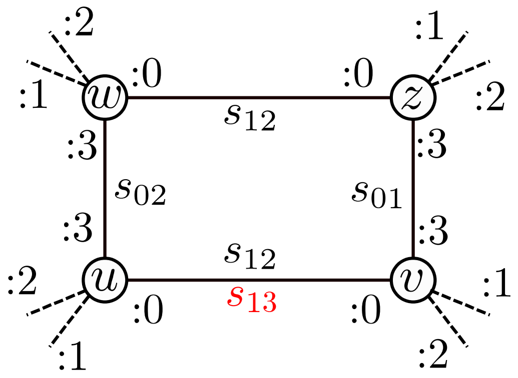

# Bounded-star complexes

To handle these problems we restrict to *bounded-star* complexes

- The *star* of a vertex *u* is the subgraph induced by *u* and its geometrical neighbors. It is denoted Star(G, u)

- A complex is *bounded-star* if all stars are bounded.

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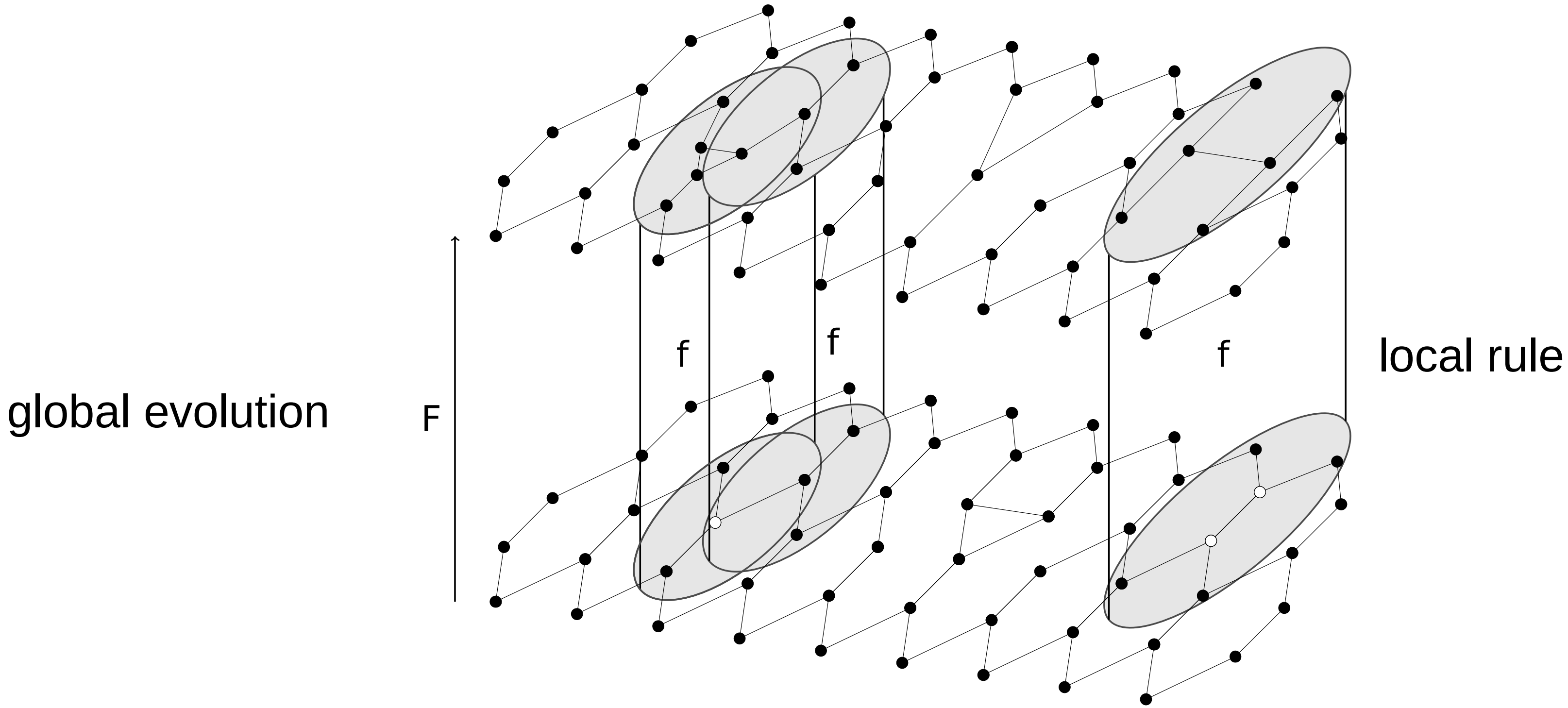
- The *star* of a vertex *u* is the subgraph induced by *u* and its geometrical neighbors. It is denoted Star(G, u)

- A complex is *bounded-star* if all stars are bounded.

➔ Graph distance is linearly bounded by Geometric distance

➔ Finite procedure to rule out torsioned complexes

# Causal Graphs Dynamics (CGD)



# Causal Graphs Dynamics: Local Rule

A function $f : D_r \to G$ is called a **local rule** if there exists some bound *b* such that:

- For all disk D and $v \in V(f(D)) \Rightarrow v \subseteq V(D).\{1, ..., b\}$.
- For all graph G and disks $D_1, D_2 \subset G$, $f(D_1)$ and $f(D_2)$ are consistent

Then, the **global evolution** is:

$$F(G) = \bigcup_v f(G_v^r)$$

where $G_v^r$ is the disk centered on *v* of radius *r*.

# Causal Dynamics of Complexes (CDC)

CDC = CGD + Rotation Commuting (NO geom. dist.)

**Proposition**

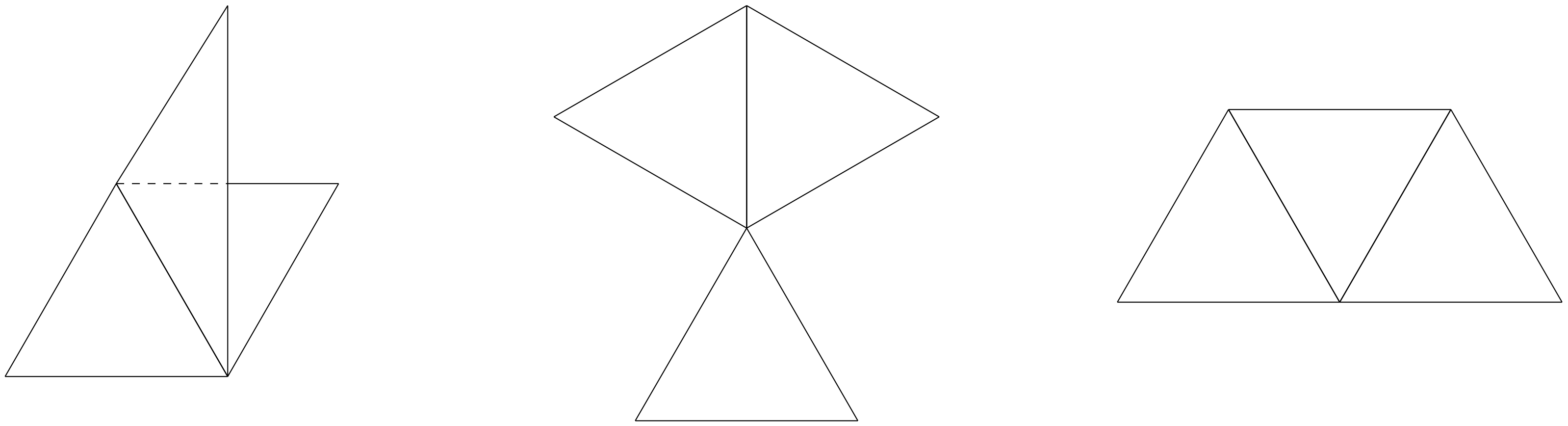
$F$ is rotation commuting if and only if there exists $f$ strongly rotation commuting

**Proposition**

It is decidable whether $f$ is strong-rotation-commuting

# Discrete Manifolds

How to single out *Discrete Manifolds* from all the graph-encoded complexes?



# Discrete Manifolds

In the continuous:

- every point of a manifold has a neighborhood homeomorphic to a ball.

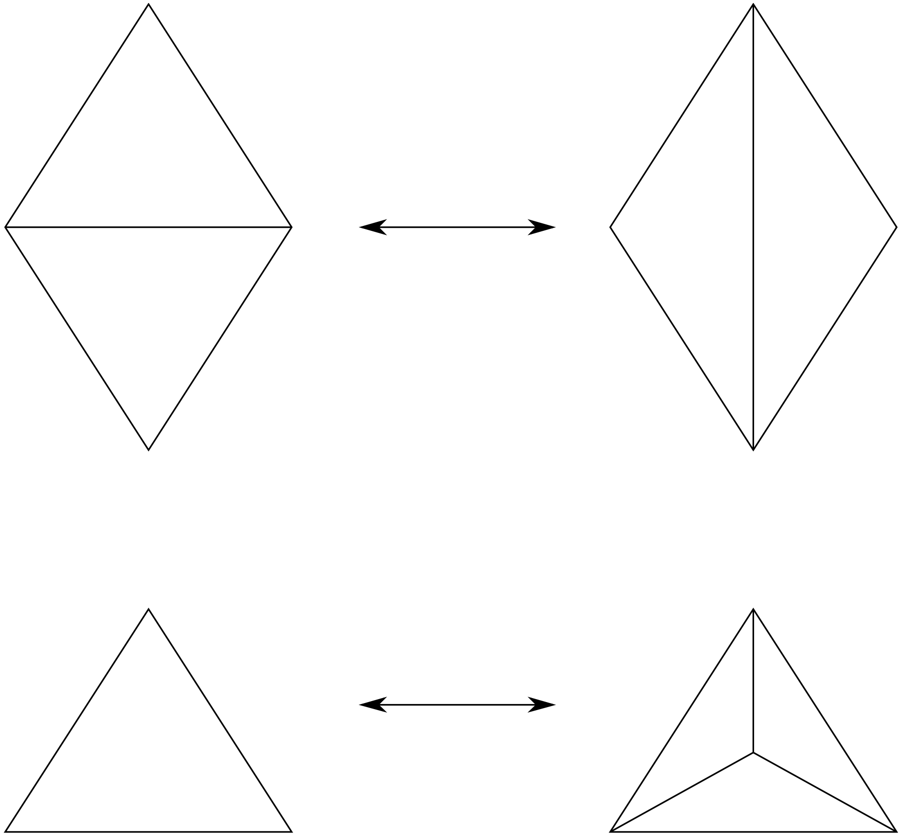
In the discrete

- first, we need to express **homeomorphisms, combinatorially**.

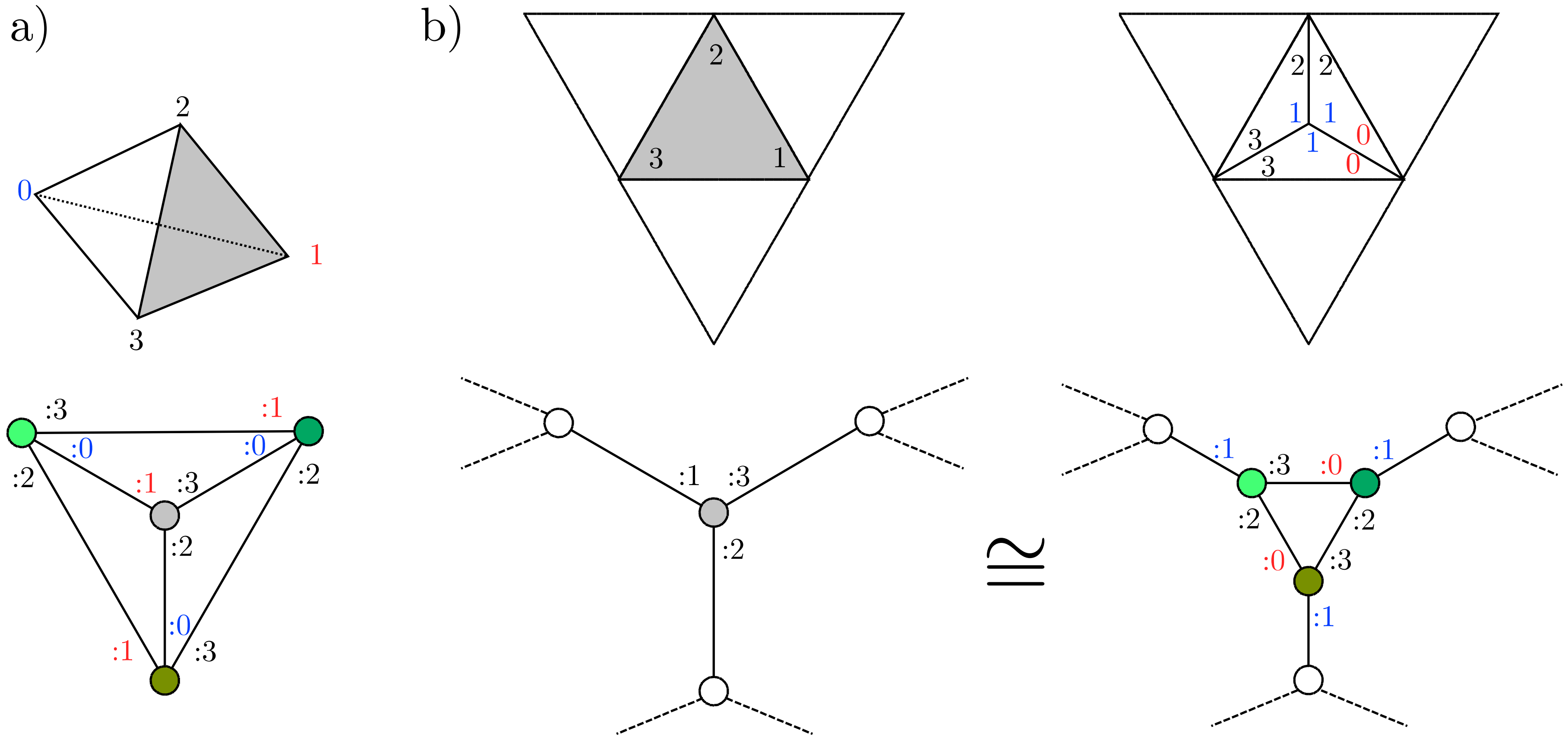
This will be done by sequences of **Pachner moves**:

- Vertex rotations
- Bistellar moves
- Graph-local (inverse) shelling

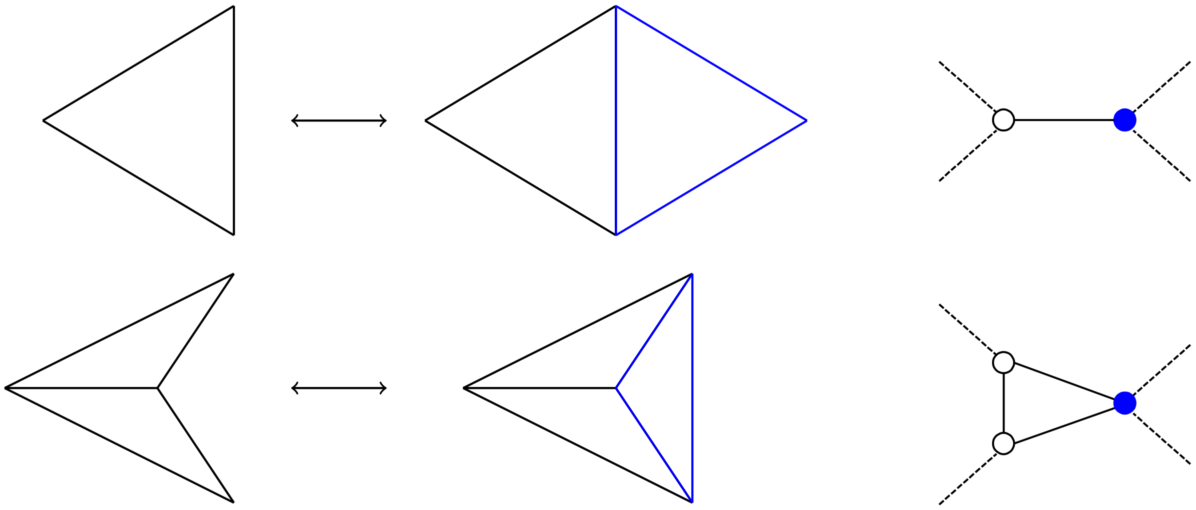
# Bistellar moves



# Bistellar moves



# (inverse) shelling



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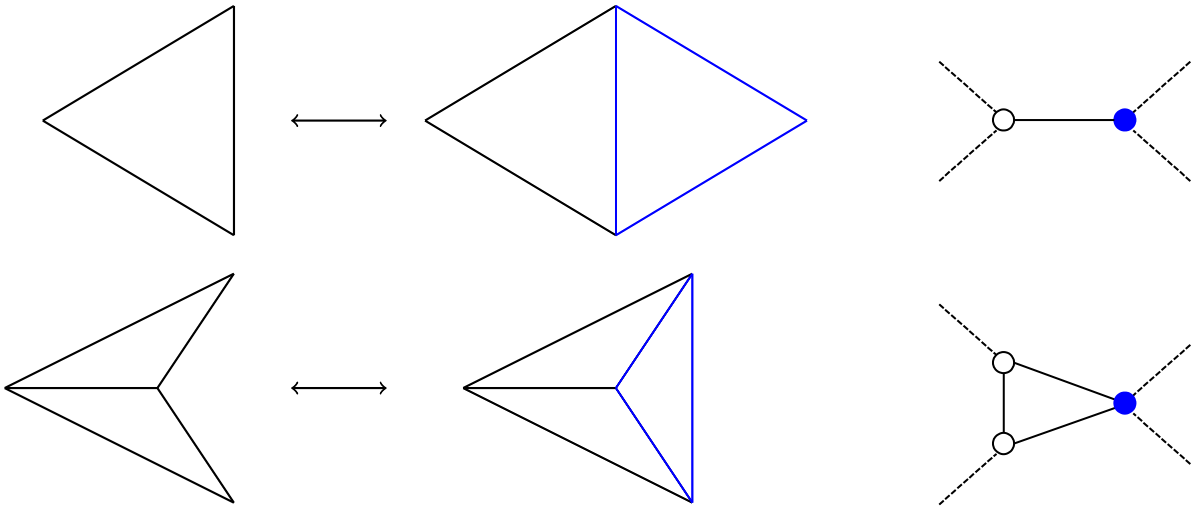
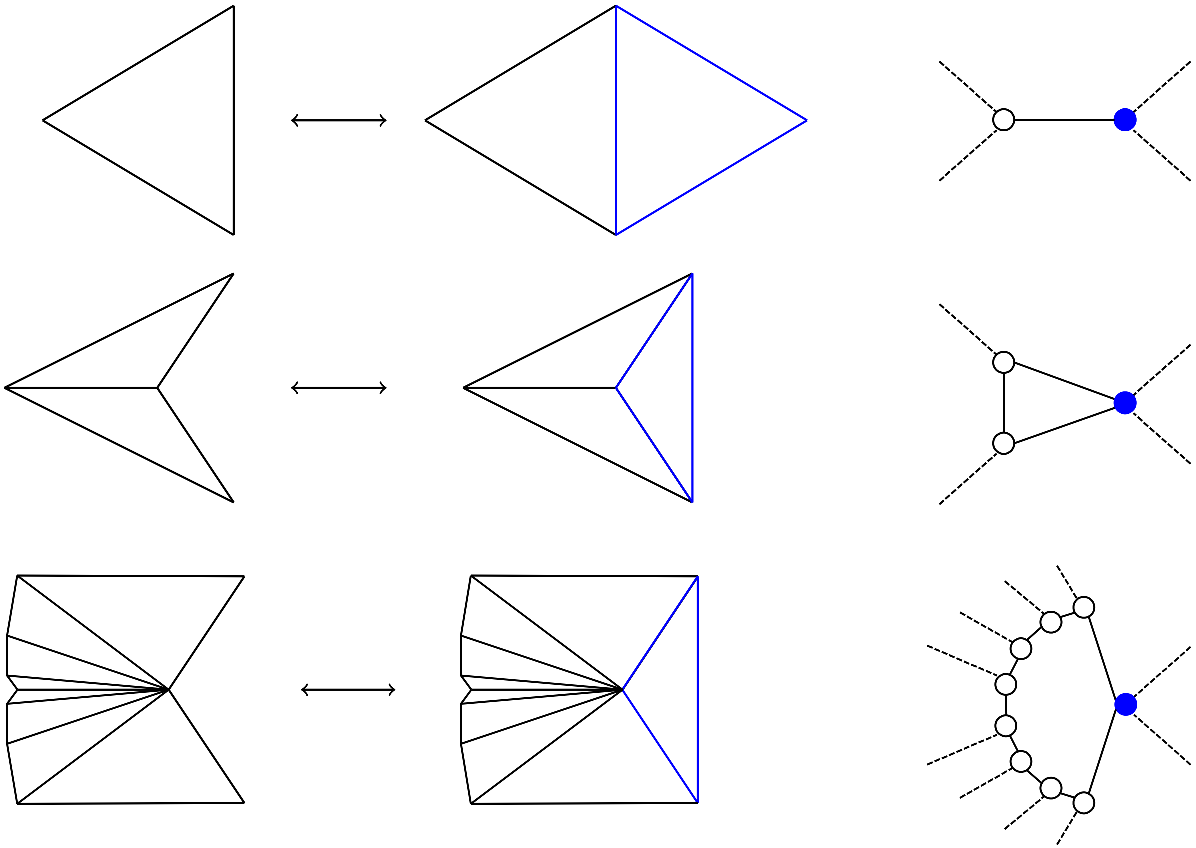


Graph-local

standard

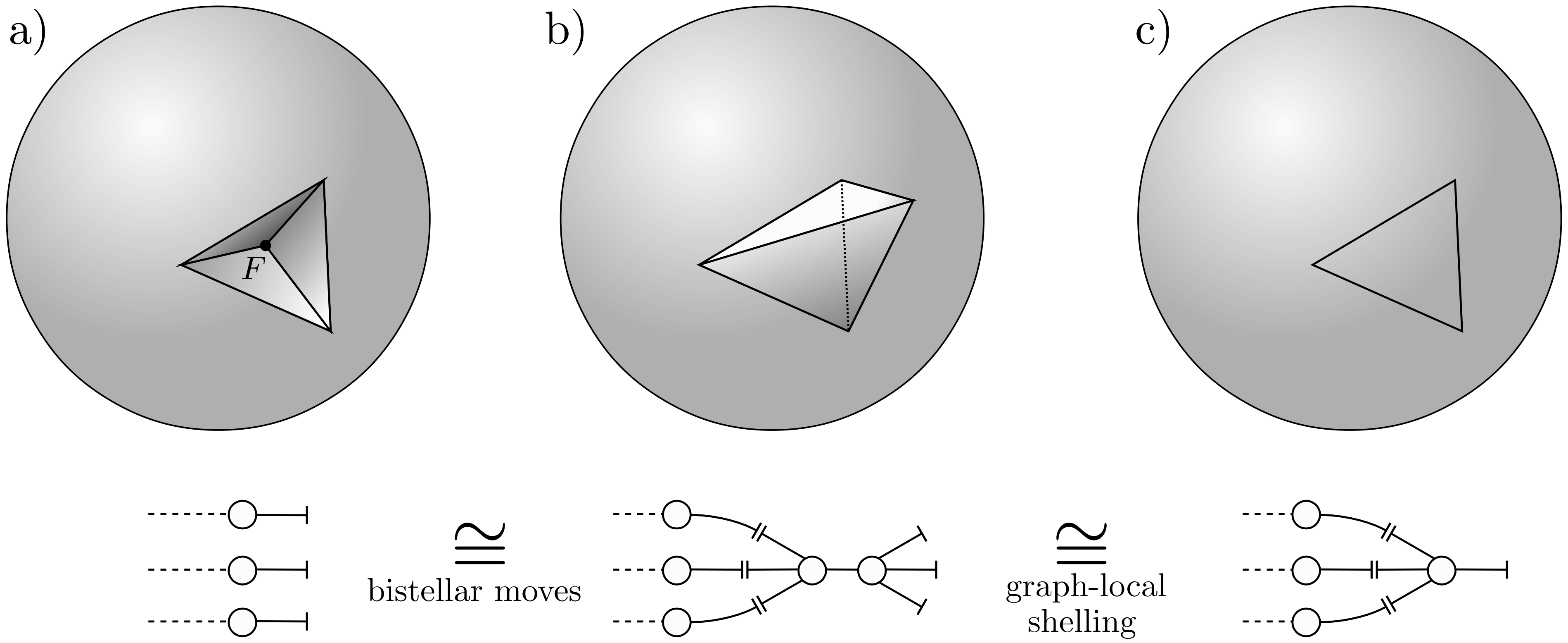
# (inverse) shelling



Graph-local

standard

# Standard (inverse) shelling = Bistellar moves + graph-local shelling



# Discrete Manifolds

**Theorem in Combinatorial Topology** [Pachner, Lickorish]

Two simplicial complexes are homeomorphic iff the are related by a sequence of standard shellings.

**Proposition**

Two simplicial complexes are homeomorphic iff the are related by a sequence of graph-local Pachner moves

**Definition**

A graph G is a **discrete manifold** if for each vertex $u$, there are Pachner moves sending Star(G, $u$) to the standard ball.

# Causal Dynamics of Discrete Manifolds

CDDM = CDC +

- Bounded-star preserving
- Torsion-free preserving
- Discrete-manifold preserving

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CDDM = CDC +

- Bounded-star preserving → **decidable**
- Torsion-free preserving
- Discrete-manifold preserving

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CDDM = CDC +

- Bounded-star preserving → decidable
- Torsion-free preserving → **decidable**
- Discrete-manifold preserving

# Causal Dynamics of Discrete Manifolds

CDDM = CDC +

- Bounded-star preserving → decidable
- Torsion-free preserving → decidable
- Discrete-manifold preserving → **decidable for $n < 4$**

# Thank you

# for your attention