Extended finite automata and decision problems for matrix semigroups

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Introduction

Aim: Make a connection between extended finite automata over matrix semigroups and decision problems for matrix semigroups
Let $G$ be a group. *Extended finite automaton over $G$, ($G$-automaton, group automaton)* is defined as
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- Register is initialized with the identity element of $G$
Let $G$ be a group. An extended finite automaton over $G$, denoted $(G\text{-}automaton, group\ automaton)$, is defined as:

- One-way finite state automaton equipped with a register
- Register is initialized with the identity element of $G$
- Register is multiplied with an element of $G$ at each step
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Let $M$ be a monoid. $M$-automaton is defined analogously.
Let $S$ be a semigroup. We want to allow the register to be multiplied elements from $S$. 
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If $S$ is not a monoid nor a group, then only the empty string can be accepted.
Let $S$ be a matrix semigroup finitely generated by a generating set of square matrices $F$. The **membership problem** is to decide whether or not a given matrix $Y$ belongs to the matrix semigroup $S$. 

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Let $S$ be a matrix semigroup finitely generated by a generating set of square matrices $F$. The **membership problem** is to decide whether or not a given matrix $Y$ belongs to the matrix semigroup $S$.

**Given:** $F = \{Y_1, Y_2, \ldots, Y_n\}$ and a matrix $Y$

**Problem:** Determine if there exist an integer $k \geq 1$ and $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$ such that $Y_{i_1} Y_{i_2} \cdots Y_{i_k} = Y$. 
When $Y$ is restricted to be the identity matrix, the problem is called the **identity problem**.
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Let $G$ be a finitely generated group and let $H$ be a subgroup of $G$. **Subgroup membership problem** or **generalized word problem** for $H$ in $G$ is to decide whether or not a given element $g \in G$ belongs to the subgroup $H$. 
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Word problem for $G$ is the membership problem for the trivial group generated by 1.
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The subgroup membership problem can be seen as a special case of the (semigroup) membership problem.
## Previous work

<table>
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<td>$\mathbb{Z}^{2\times2}$</td>
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<td>$H$</td>
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The emptiness problem and subgroup membership problem

**Theorem**

Let $H$ be a finitely generated subgroup of $G$. If the emptiness problem for $G$-automata is decidable, then the subgroup membership problem for $H$ in $G$ is decidable.
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Proof.

- Let $g$ be an element from $G$. We should decide whether $g \in H$. 
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- Let $g$ be an element from $G$. We should decide whether $g \in H$.
- Construct $G$-automaton $V$.
- We are going to show that $g \in H$ iff $L(V)$ is nonempty.
The emptiness problem and subgroup membership problem

Proof.

\{h_1, \ldots, h_n\} generates $H$

Claim: $g \in H$ iff $L(V)$ is nonempty.
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- \( \{h_1, \ldots, h_n\} \) generates \( H \)

\[ q_1 \xrightarrow{a, g} q_2 \xrightarrow{a, h_i} \]
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- $h_{i_1} h_{i_2} \cdots h_{i_k} = g^{-1}$ for some $k \geq 1$ and $i_1, i_2, \ldots, i_k \in \{1, \ldots, n\}$
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The emptiness problem and subgroup membership problem

Proof.

- Suppose $L(V)$ is nonempty
- Acceptance condition: register is equal to identity
- Register is initially multiplied by $g \implies H$ contains $g^{-1}$.
- $g^{-1} \in H \implies g \in H$
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Suppose that the emptiness problem for $G$-automaton is decidable.
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  - Construct $V$
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$\implies$ subgroup membership problem for $H$ is decidable.
Given a matrix $Y$ from $\mathbb{Z}^{2\times 2}$ and a subgroup $H$ of $\mathbb{Z}^{2\times 2}$, it is decidable whether $Y$ belongs to $H$. 
Decidability of the subgroup membership problem for $\mathbb{Z}^{2\times 2}$

**Theorem**

Given a matrix $Y$ from $\mathbb{Z}^{2\times 2}$ and a subgroup $H$ of $\mathbb{Z}^{2\times 2}$, it is decidable whether $Y$ belongs to $H$.

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- We will prove that the emptiness problem for $\mathbb{Z}^{2\times 2}$-automata is decidable.
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- We will prove that the emptiness problem for $\mathbb{Z}^{2\times 2}$-automata is decidable.
- Suppose that a $\mathbb{Z}^{2\times 2}$-automaton $V$ is given.
  - Remove edges labeled by a non-invertible matrix from $V$.
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- Suppose that a $\mathbb{Z}^{2 \times 2}$-automaton $V$ is given.
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  - Matrices multiplied by the register are invertible and belong to $GL(2, \mathbb{Z})$. 

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- Suppose that a $\mathbb{Z}^{2 \times 2}$-automaton $V$ is given.
  - Remove edges labeled by a non-invertible matrix from $V$.
  - Matrices multiplied by the register are invertible and belong to $GL(2, \mathbb{Z})$.
- $V$ is a $GL(2, \mathbb{Z})$-automaton.
Decidability of the subgroup membership problem for $\mathbb{Z}^{2 \times 2}$

Lemma

Let $G$ be a finitely generated group and let $H$ be a subgroup of finite index. Any $G$-automaton can be converted into an $H$-automaton recognizing the same language.
Decidability of the subgroup membership problem for $\mathbb{Z}^{2\times2}$

**Lemma**

Let $G$ be a finitely generated group and let $H$ be a subgroup of finite index. Any $G$-automaton can be converted into an $H$-automaton recognizing the same language.

**Lemma**

Any $F_2$-automaton can be converted into a pushdown automaton recognizing the same language.
Decidability of the subgroup membership problem for $\mathbb{Z}^{2\times2}$

**Theorem**

Given a matrix $Y$ from $\mathbb{Z}^{2\times2}$ and a subgroup $H$ of $\mathbb{Z}^{2\times2}$, it is decidable whether $Y$ belongs to $H$.

**Proof.**

- A pushdown automaton recognizing $L(V)$ can be constructed
  - $F_2$ has finite index in $GL(2, \mathbb{Z})$
  - $F_2$-automaton recognizing $L(V)$ can be constructed
  - $F_2$-automaton can be converted to a pushdown automaton

- Emptiness problem for pda is decidable $\implies$ Emptiness problem for $\mathbb{Z}^{2\times2}$-automata is decidable
The emptiness problem and identity problem

**Theorem**

Let $S$ be a semigroup. The identity problem for $S$ is decidable if the emptiness problem for $S$-automaton is decidable.
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Let $S$ be a semigroup. The identity problem for $S$ is decidable if the emptiness problem for $S$-automaton is decidable.

**Proof.**

- Construct an $S$-automaton $V$ such that $1 \in S$ iff $L(V)$ is nonempty.
Undecidability of the emptiness problem for $\mathbb{Z}^{4\times 4}$-automata

**Fact**

Given a semigroup $S$ generated by eight $4 \times 4$ integer matrices, determining whether the identity matrix belongs to $S$ is undecidable. [KNP17]
Undecidability of the emptiness problem for $\mathbb{Z}^{4 \times 4}$-automata

**Fact**

Given a semigroup $S$ generated by eight $4 \times 4$ integer matrices, determining whether the identity matrix belongs to $S$ is undecidable. [KNP17]

**Corollary**

Let $S$ be a subsemigroup of $\mathbb{Z}^{4 \times 4}$ generated by eight matrices. The emptiness problem for $S$-automaton is undecidable.
Fact

Given a semigroup $S$ generated by eight $4 \times 4$ integer matrices, determining whether the identity matrix belongs to $S$ is undecidable. [KNP17]

Corollary

Let $S$ be a subsemigroup of $\mathbb{Z}^{4\times4}$ generated by eight matrices. The emptiness problem for $S$-automaton is undecidable.

Proof.

We know that the identity problem for $S$ is undecidable. By the above theorem the result follows.
Theorem

Let $H$ be a finitely generated subgroup of $G$. If the universe problem for $G$-automata is decidable, then the subgroup membership problem for $H$ in $G$ is decidable.
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Let $H$ be a finitely generated subgroup of $G$. If the universe problem for $G$-automata is decidable, then the subgroup membership problem for $H$ in $G$ is decidable.

Proof.

- Construct $G$-automaton $V$ such that $g \in H$ iff $L(V) = \Sigma^*$.
Theorem

Let $S$ be a finitely generated semigroup. If the universe problem for $S$-automata is decidable, then the identity problem for $S$ is decidable.
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Remark

Converses are not true:
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  - For a given pushdown automaton, an $F_2$-automaton recognizing the same language can be constructed
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Converses are not true:

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  - For a given pushdown automaton, an $F_2$-automaton recognizing the same language can be constructed
  - Universe problem for pushdown automata is undecidable
Remark

Converses are not true:

- **Universe problem for** $F_2$-automaton is undecidable
  - For a given pushdown automaton, an $F_2$-automaton recognizing the same language can be constructed
  - Universe problem for pushdown automata is undecidable

- $F_2$ is a subgroup of $SL(2, \mathbb{Z})$ and the membership and identity problems for $SL(2, \mathbb{Z})$ are decidable
Conclusion

We make a connection between the decidability of the subgroup membership and identity problems and the universe and emptiness problems for extended finite automata.
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- Emptiness problem for $S$-automata
  - Decidability of the subgroup membership problem for $\mathbb{Z}^{2\times2}$
  - Undecidability of the emptiness problem for $\mathbb{Z}^{4\times4}$-automata

- Universe problem for $S$-automata
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We make a connection between the decidability of the subgroup membership and identity problems and the universe and emptiness problems for extended finite automata

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  - Undecidability of the emptiness problem for $\mathbb{Z}^{4\times4}$-automata
- Universe problem for $S$-automata

Identity and membership problems for $3 \times 3$ integer matrix groups are open.
Thank You!