ON REGULAR EXPRESSIONS WITH BACKREFERENCES AND TRANSDUCERS

Frank Drewes

Martin Berglund    Brink van der Merwe
Regular expressions with backreferences

\textbf{REGULAR EXPRESSION} \((.+)_1\)
Regular expressions with backreferences

\texttt{REGULAR EXPRESSION} \( (.+)_1 \)
Regular expressions with backreferences

```
(.+)

(?<first>.+)
```

REGULAR EXPRESSION
Regular expressions with backreferences

\[(.+)_1\]

\[('first').+\]k'first'

**Test String**

**Regular Expression**

abab
Regular expressions with backreferences

**REGULAR EXPRESSION**

```
(.+)\1
(?<first>.+)\k[first]
```

**TEST STRING**

```
abab
```

**MATCH INFORMATION**

- Full match: 0-4 `abab`
- Group 1: 0-2 `ab`
REGULAR EXPRESSION

\[[0-9]+(\d*)\1+\]
REGULAR EXPRESSION

\[[0-9]+\.[0-9]*\(\d+\)\]1+

TEST STRING

0.818181
REGULAR EXPRESSION

\[0-9]+\.(d*)(d+)\1+\]

TEST STRING

0.818181

MATCH INFORMATION

Full match 0-8 `0.818181`
Group 1. 4-6 `81`
Your regular expression does not match the subject string.
But how do we handle the following example:

In most matching engines the subexpression (?i) matches the empty string, but enables case-insensitive matching.

Thus (?i)(.*)\1 matches any \(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n\) where, \(\alpha_i\) and \(\beta_i\) are the same letter up to one (perhaps) being lowercase and the other uppercase.
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In most matching engines the subexpression (?i) matches the empty string, but enables case-insensitive matching.

Thus (?i)(.*)\1 matches any $\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_n$ where, $\alpha_i$ and $\beta_i$ are the same letter up to one (perhaps) being lowercase and the other uppercase.

We permit transducer subexpressions, obtained by allowing the application of some string-to-string transducer to subexpressions.
A transducer subexpression $t(E)$ describes the language of strings obtained by applying the transducer $t$ to the language matched by $E$. 
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We call these extended expressions, obtained by adding backreferences and transducers, regular expressions with backreferences and transducers (REbt).
A simple class of transducers over $\Sigma^*$, corresponding to transducers with only one state, is the set of all $t = (\alpha_1: \beta_1, \ldots, \alpha_k: \beta_k)$ where $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in \Sigma \cup \{\varepsilon\}$.

The transduction denoted by $t$ is

$$\mathcal{L}(t) = \{(\alpha_{i_1} \cdots \alpha_{i_n}, \beta_{i_1} \cdots \beta_{i_n}) \mid n \in \mathbb{N}\}.$$
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The transduction denoted by $t$ is

$$L(t) = \{(\alpha_i \cdot \alpha_n, \beta_i \cdot \beta_n) \mid n \in \mathbb{N}\}.$$

If $t_b = a : b$, $t_c = a : c$ and $t_d = b : d$ then

- $L([1a^*]_1 t_b(\uparrow 1)t_c(\uparrow 1)) = \{a^n b^n c^n \mid n \in \mathbb{N}\}$

- $L([1a^*]_1[2b^*]_1 t_c(\uparrow 1)t_d(\uparrow 2)) = \{a^m b^n c^m d^n \mid m, n \in \mathbb{N}\}$
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Theorem

For a recursively enumerable language $L$ there exists an expression $E$ with backreferences and transducers (i.e. $E \in \text{REbt}$), such that $\mathcal{L}(E) = L$. Consequently the membership problem is undecidable for \text{REbt}.
The Bad News

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Proof

For Turing machine $M$ we construct the following transducers:
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For Turing machine \( M \) we construct the following transducers:

- a transducer \( t_{\text{init}} \) such that \((w, c) \in \mathcal{L}(t_{\text{init}}) \) if \( c \in \Sigma^* \) is the initial configuration of \( M \) when starting with \( w \) as input,
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- a transducer \( t_{\text{acc}} \) such that \( (\cdot, c) \in \mathcal{L}(t_{\text{acc}}) \) if \( c \) is the concatenation of configurations of \( M \), with only the last configuration being accepting, and
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- a transducer $t_{\text{step}}$ such that $(c, c') \in \mathcal{L}(t_{\text{step}})$ if $M$ can go from the configuration $c$ to the configuration $c'$ in a single step.
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\[
E = \left[ \phi \Gamma^* \right]_\phi D([\phi t_{\text{init}}(\uparrow_\phi)]\phi t_{\text{acc}}([\phi t_{\text{step}}(\uparrow_\phi)]\phi^*)) \text{, where } D \text{ is a transducer that deletes the entire input, and outputs } \varepsilon.
\]

The first subexpression selects and captures any input string $w$. The subexpression $D(\cdots)$ simulates a computation of $M$ on $w$ to either fail or, if $M$ accepts, yield $\varepsilon$.  

The Bad News

Theorem

For a recursively enumerable language $L$ there exists $E \in \text{REbt}$ such that $L(E) = L$. Consequently the membership problem is undecidable for REbt.

Proof

For a Turing machine $M$ with input alphabet $\Gamma$, choose a presentation of the configurations of $M$ as strings $w \in \Gamma^*$, where $\Gamma \subseteq \Sigma$.

We construct

- a transducer $t_{\text{init}}$ such that $(w, c) \in L(t_{\text{init}})$ if $c \in \Gamma^*$ is the initial configuration of $M$ when starting with $w \in \Sigma^*$ as input,
- a transducer $t_{\text{acc}}$ such that $(c,c) \in L(t_{\text{acc}})$ if $c$ is the concatenation of configurations of $M$, with only the last configuration being accepting, and
- a transducer $t_{\text{step}}$ such that $(c,c_0) \in L(t_{\text{step}})$ if $M$ can go from the configuration $c$ to the configuration $c_0$ in a single step.

$E = [\phi \Gamma^*]_\phi \begin{array}{c} D([\phi t_{\text{init}}(\uparrow_\phi)]_\phi t_{\text{acc}}([\phi t_{\text{step}}(\uparrow_\phi)]_\phi *)) \end{array}$, where $D$ is a transducer that deletes the entire input, and outputs $\varepsilon$.

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For a recursively enumerable language \( L \) there exists an expression \( E \) with backreferences and transducers (i.e. \( E \in \text{REbt} \)), such that
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Consequently the membership problem is undecidable for \( \text{REbt} \).

Proof

For a Turing machine \( M \) with input alphabet \( \Gamma \), choose an representation of the configurations of \( M \) as strings \( w \in \Gamma^* \), where \( \Gamma \in \Sigma \).

We construct
- \( \text{transducer} \ t_{\text{init}} \) such that \((w, c) \in L(t_{\text{init}})\) if \( c \in \Gamma^* \) is the initial configuration of \( M \) when starting with \( w \in \Gamma^* \) as input,
- \( \text{transducer} \ t_{\text{acc}} \) such that \((c, c) \in L(t_{\text{acc}})\) if \( c \) is the concatenation of configurations of \( M \), with only the last configuration being accepting, and
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Consequently the membership problem is undecidable for REbt.

Proof

For a Turing machine \( M \) with input alphabet \( \Gamma \), choose a representation of the configurations of \( M \) as strings \( w \in \Gamma \star \Gamma \), where \( \Gamma \star \Gamma \).

We construct

- a transducer \( \text{init} \) such that \((w, c) \in L(\text{init})\) if \( c \in \Gamma \star \Gamma \) is the initial configuration of \( M \) when starting with \( w \in \Gamma \star \) as input,
- a transducer \( \text{acc} \) such that \((c, c) \in L(\text{acc})\) if \( c \) is the concatenation of configurations of \( M \), with only the last configuration being accepting, and
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E = [\phi \Gamma^*]_\phi D([\phi \text{init}(\uparrow_\phi)]_\phi \text{acc}([\phi \text{step}(\uparrow_\phi)]^*_\phi)),
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where \( D \) is a transducer that deletes the entire input, and outputs \( \varepsilon \).
Consider various restrictions

- Allow backreferences but not transducers.

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- Do not allow the capture of transducer preimages.

- Allow only non-deleting transducers.

- Do not allow transducers in capturing cycles – a capturing cycle captures a submatch and then later backreferences this capture as part of another submatch by the same capturing subexpression.

- Allow only functional transducers.

- Allow only a single top-level transducer.
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\[E = [\phi \Gamma^*]_\phi D([\phi t_{init}(\uparrow_\phi)]_\phi t_{acc}(\phi t_{step}(\uparrow_\phi))_{\phi^*})\], where \(D \in \text{FST}\) deletes the entire input and outputs \(\varepsilon\).
Theorem

For expressions $E$ without transducers we may decide whether $w \in \mathcal{L}(E)$ in time polynomial in $|w|$ and $|E|$, with the degree of the polynomial a constant times the number of backreference symbols in $E$. 
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Automata $\{A_1, \ldots, A_n\}$ – Question: $\mathcal{L}(A_1) \cap \cdots \cap \mathcal{L}(A_n) = \emptyset$.
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The intersection is non-empty if and only if $\varepsilon \in \mathcal{L}(E)$. 
Theorem

Uniform membership for the class of expressions $E$ with backreferences and non-deleting transducers is EXPSPACE-complete.
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Theorem

The uniform membership problem for the class of expressions with only a top-level transducer is PSPACE-complete.

For the subclass of expressions where the top-level transducer is also non-deleting, the uniform and non-uniform membership problems are NP-complete.
We have:

(i) proposed an extension of regular expressions with backreferences, by adding transducers;

(ii) established that this makes membership testing intractable; and

(iii) explored various restrictions to form a practical basis for use in software.

Future work

The precise expressiveness of the classes should be considered – several gaps exist beyond what follows naturally from what we have done here. The subclasses should also be compared with respect to succinctness, and there remain some open questions regarding computational complexity of for example (uniform) membership in certain cases.
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[2b | !2b]

That is the expression.