Relations preserving the convergence of series in topological groups

Peter Eliaš

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Introduction

Theorem (R. Rado, 1960)

Let $f : \mathbf{R} \to \mathbf{R}$. *TFAE:*

1. f *preserves the convergence of series, i.e., for every* $\{x_n\}_{n\in\mathbf{N}}$, *if* $\sum x_n$ *converges then* $\sum f(x_n)$ *converges,*
2. $f(x) = ax$ *holds on a neighborhood of* 0.

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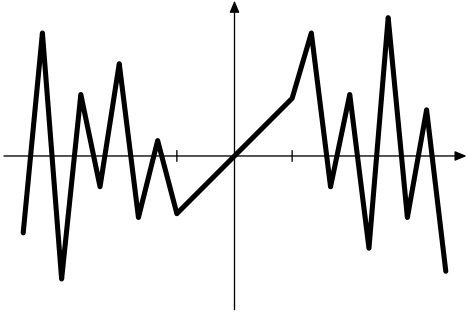

The inverse problem

Problem (J. Borsík)

Characterize the functions $f : \mathbf{R} \to \mathbf{R}$ preserving the divergence of series, i.e., such that $\sum f(x_n)$ is divergent for every divergent $\sum x_n$.

J. Borsík, *Functions preserving some types of series*, presented on 20$^{\text{th}}$ Summer Conference of Real Functions Theory, Liptovský Ján, 2006.

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Typical functions preserving the divergence of series are

- $f(x) = ax$, for some $a \neq 0$,
- $f(x) \geq a|x|$, for some $a > 0$.

Solution of the inverse problem

Theorem (P. Eliaš)

Function $f : \mathbf{R} \to \mathbf{R}$ *preserves the divergence of series iff there exist* $a \neq 0$, $c > 0$ *such that either*

1. $\forall x \ (x = 0 \vee |f(x)| \geq c \vee f(x) = ax)$, *or*
2. $\forall x \left(x = 0 \vee |f(x)| \geq c \vee \dfrac{f(x)}{a} \geq |x|\right)$.

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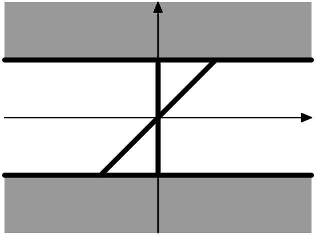
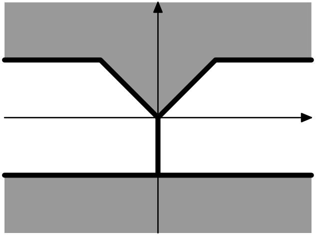
The conditions above mean that the graph of f is included in one of the following sets (displayed is the case $a > 0$):





Sketch of the proof

$\Leftarrow$: Let $a \neq 0$, $c > 0$ be such that either

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Then f preserves the divergence of series (*an easy exercise*).

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$\Rightarrow$: Assume that f is divergence preserving. For $d > 0$, define $P_d = f[(0, \infty)] \cap (-d, d)$, $N_d = f[(-\infty, 0)] \cap (-d, d)$.

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- There is $d > 0$ such that P_d, N_d are sign-homogeneous. Otherwise one can find $x_n > 0$ such that $f(x_n) \to 0$ and the signs of $f(x_n)$ are alternating.

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- If for all $d > 0$ the sets P_d, N_d are nonempty and their signs are opposite then there are $a \neq 0$, $c > 0$ satisfying (1).
- If the signs of P_d, N_d are equal (or one of the sets is empty) then there are $a \neq 0$, $c > 0$ satisfying (2).

Possibilities of generalization

We shall try to generalize these results in two ways:

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We shall try to generalize these results in two ways:

1. consider topological groups G, H and functions $f : G \longrightarrow H$ instead of $f : \mathbf{R} \longrightarrow \mathbf{R}$,
2. consider binary relations instead of functions.

Topological groups

$(G, \cdot, \mathcal{O})$ is a topological group if

- $(G, \cdot)$ is a group,
- $(G, \mathcal{O})$ is a topological space, and
- the group operations $\cdot$, $^{-1}$ are continuous.

We denote by e_G the neutral element of G.

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Examples:

- $(\mathbf{R}, +)$, $(\mathbf{R}^n, +)$ do not have arbitrarily small subgroups,
- $(\{0, 1\}^{\mathbf{N}}, +)$, $(\mathbf{R}^{\mathbf{N}}, +)$ have arbitrarily small subgroups,
- infinite products of topological groups equipped with product topology have arbitrarily small subgroups.

Preserving the convergence in topological groups

Definition

A sequence $\{x_n\}_{n\in\mathbf{N}}$ is called

- Cauchy if for every neighborhood U of e_G there is N such that if $m, n > N$ then $x_m^{-1}x_n \in U$.
- Cauchy multipliable if the sequence of products $\left\{\prod_{n<k} x_n\right\}_{k\in\mathbf{N}}$ is a Cauchy sequence.

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Definition (W. Freedman)

Let G, H be topological groups.
A function $f : G \to H$ is convergence preserving if it maps every Cauchy multipliable sequence to a Cauchy multipliable sequence.

W. Freedman, *Convergence preserving mappings in topological groups*, Topol. Appl. **154** (2007), 1089–1096.

Results of Freedman and Shibakov

Definition

A function $f : G \to H$ is called

- sequentially continuous at x if $x_n \to x$ implies $f(x_n) \to f(x)$,
- local sequential homomorphism if it is sequentially continuous at e_G and $x_n \to e_G \ \wedge \ y_n \to e_G$ implies $\exists N \ \forall n > N \ f(x_n y_n) = f(x_n) f(y_n)$.

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Theorem (W. Freedman, A. Shibakov)

Let G, H be Hausdorff topological groups such that H does not have arbitrarily small subgroups. TFAE:

1. *f is convergence preserving,*
2. *f is local sequential homomorphism.*

The case of H having arbitrarily small subgroups

Definition

A function $f : G \to H$ is called local sequential pseudo-homomorphism if

1. it is sequentially continuous at e_G, and
2. if $x_n \to e_G$, $y_n \to e_G$ then for every neighborhood U of e_H there is N such that the group generated by the set $\{f(x_n y_n)^{-1} f(x_n) f(y_n) : n > N\}$ is included in U.

Clearly if H does not have arbitrarily small subgroups then f is LSPH iff it is LSH.

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Definition

A sequence of subgroups $\{H_n\}_{n \in \mathbf{N}}$ of a group H is called a chain, if

1. every H_{n+1} is a subgroup of H_n, and
2. for every neighborhood U of e_H there is n such that $H_n \subseteq U$.

The main result

Theorem (P. Eliaš)

Let G, H be arbitrary Hausdorff topological groups. TFAE:

1. *f is convergence preserving,*
2. *f is LSPH,*
3. *for any sequences $x_n \longrightarrow e_G$, $y_n \longrightarrow e_G$ there is a chain $\{H_n\}_{n\in\mathbf{N}}$ such that $\forall^\infty n\ f(x_n y_n)^{-1} f(x_n) f(y_n) \in H_n$.*

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Problem

Characterize the functions $f : G \longrightarrow H$ which are divergence preserving, i.e., such that $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy multipliable whenever $\{f(x_n)\}_{n\in\mathbf{N}}$ is.

Relations preserving the convergence and divergence

Definition

Let G, H be topological groups. We say that $R \subseteq G \times H$ is

- convergence preserving if $\{y_n\}_n$ is Cauchy multipliable whenever $\{x_n\}_n$ is Cauchy multipliable and $\forall n \ (x_n, y_n) \in R$,

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3. If $R \subseteq G \times H$ is CP then $R \cup (G \times \{e_H\})$ is CP.
4. If G has a countable base of neighborhoods of e_G then the topological closure of a CP relation is CP.

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Problem

Is every CP relation a subset of $S \cup ((G \setminus U) \times H) \cup (G \times \{e_H\})$ for some sequentially closed subsemigroup S and a neighborhood U of e_G?