$\mathcal{F}$-additive sets for some families of thin sets

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Let $\mathbb{T}$ be the circle group, i.e. the set $[0,1)$ with operations carried out modulo integers.
For $x \in \mathbb{T}$, let $\|x\|$ be the distance of x to 0, thus $\|x\| \in \left[0, \frac{1}{2}\right]$.
$\mathbb{T}$ is a compact Polish space.

Definition A set $X \subseteq \mathbb{T}$ is:

A-set if there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\|n_k x\| \to 0$ on X,

D-set if there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\|n_k x\| \rightrightarrows 0$ on X,

N-set if there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ of non-negative reals such that $\sum a_n = \infty$ and $\sum a_n \|nx\| < \infty$ on X,

wD-set if there is a Borel set $Y \supseteq X$ such that for any continuous Borel measure μ on $\mathbb{T}$ there is an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \int_Y \|n_k x\| \,\mathrm{d}\mu(x) = 0.$$

Let $\mathcal{A}$, $\mathcal{D}$, $\mathcal{N}$, $w\mathcal{D}$ denote the families of all A-, D-, N-, and wD-sets, respectively.

Basic properties

1. Families $\mathcal{A}$, $\mathcal{D}$, $\mathcal{N}$, $w\mathcal{D}$ are closed under translations and taking subsets, have Borel bases ($\mathcal{D}$ – closed, $\mathcal{N} - F_\sigma$, $\mathcal{A} - G_{\delta\sigma}$);
2. $\mathcal{D} \subseteq \mathcal{A} \subseteq w\mathcal{D}$, $\mathcal{D} \subseteq \mathcal{N} \subseteq w\mathcal{D}$, $\mathbb{T} \notin w\mathcal{D}$;
3. a subgroup of $\mathbb{T}$ generated by an A-set (N-set, wD-set) is again an A-set (N-set, wD-set);
4. there exist D-sets X, Y such that $X + Y = \mathbb{T}$, thus $X \cup Y \notin w\mathcal{D}$.

Definition (Arbault 1952) A set X is *$\mathcal{N}$-permitted* if for any N-set Y, $X \cup Y$ is an N-set.

Theorem (Arbault, Erdös) Any countable set is $\mathcal{N}$-permitted.

Problem (Bary 1961) Does there exist a perfect $\mathcal{N}$-permitted set?

Theorem (Lafontaine 1969) Any closed $\mathcal{N}$-permitted set is countable.

Theorem (P.E. 2004) Any $\mathcal{A}$-permitted set is perfectly meager, i.e. meager relatively to any perfect set.

Definition Let $\mathcal{F}$ be a family of sets. A set X is called *$\mathcal{F}$-additive* if for any $Y \in \mathcal{F}$, $X + Y \in \mathcal{F}$.

Remark A set is $\mathcal{A}$-permitted ($\mathcal{N}$-, $w\mathcal{D}$-permitted) iff it is $\mathcal{A}$-additive, ($\mathcal{N}$-, $w\mathcal{D}$-additive).

Remark Lafontaine actually showed that for any perfect set $P \subseteq \mathbb{T}$ there exists D-set D such that $P + D$ has positive Lebesgue measure.

Theorem (Erdös, Kunen, Mauldin 1981) For any perfect set $P \subseteq \mathbb{R}$ there exists a perfect set Q having Lebesgue measure zero such that $P + Q = \mathbb{R}$.

Theorem For any perfect set $P \subseteq \mathbb{T}$ there exists D-set D such that $P + D = \mathbb{T}$.

Proof Let $B_k = \{m2^{-k-1} : m \in 2^{k+1}\}$. By induction for $k \in \mathbb{N}$ define n_k, $\varepsilon_k > 0$, and $A_k \in [P]^{<\omega}$ such that for every $a \in A_k$ and $b \in B_k$ there is $a' \in A_{k+1}$ such that

1. $\|a' - a\| < \varepsilon_k/2$,
2. if $\|x - a'\| < \varepsilon_{k+1}$ then $\left\|n_{k+1}x - b\right\| < 2^{-k-2}$.

Put $D = \{x : \forall k \ \ \|n_k x\| \leq 2^{-k}\}$. For a given $y \in \mathbb{T}$, let $b_k \in B_k$ be such that $\left\|n_{k+1}y - b_k\right\| \leq 2^{-k-2}$.
By induction choose $p_k \in A_k$ such that $p_k \to p$ and for all k, $\left\|n_{k+1}p - b_k\right\| \leq 2^{-k-2}$.
Then $p \in P$ and $y - p \in D$, hence $y \in P + D$.

Corollary If $\mathcal{F}$ is a family of subsets of $\mathbb{T}$ such that $\mathcal{D} \subseteq \mathcal{F}$ and $\mathbb{T} \notin \mathcal{F}$ then any $\mathcal{F}$-additive set is totally imperfect.

Remark For a given perfect set P, we have defined a set $D = \{x : \forall k \;\; \|n_k x\| \leq 2^{-k}\}$ so that for any $y \in \mathbb{T}$, $P \cap (y - D) \neq \emptyset$.
We can moreover arrange that for all $y \in \mathbb{T}$,

1. $P \cap (y - D)$ contains a perfect subset,
2. if $E = \{x : \forall^\infty k \;\; \|n_k x\| \leq 2^{-k}\}$ then $P \cap (y - E)$ is dense in P.

Definition A set $X \subseteq \mathbb{T}$ is s_0*-set* if every perfect set $P \subseteq \mathbb{T}$ has a perfect subset Q disjoint with X.

Theorem Let $\mathcal{D} \subseteq \mathcal{F} \subsetneq \mathcal{P}(\mathbb{T})$. Then any $\mathcal{F}$-additive set is s_0-set.
Proof Let X be an $\mathcal{F}$-additive, not s_0-set. There is a perfect set P such that for any perfect $Q \subseteq P$, $Q \cap X \neq \emptyset$. Find a D-set D as above. Since X is $\mathcal{F}$-additive, there exists $y \in \mathbb{T} \setminus (X + D)$. We have the set D such that $P \cap (y - D)$ has a perfect subset, but this set must intersect X. But if $x \in X \cap (y - D)$ then $y \in x + D$, a contradiction.

Definition (Nowik, Scheepers, Weiss) A set $X \subseteq \mathbb{T}$ is *perfectly meager in transitive sense* if for every perfect set $P \subseteq \mathbb{T}$ there exists an F_σ set $F \supseteq X$ such that for all $y \in \mathbb{T}$, $P \cap (y + F)$ is meager in P.

Notation Let $\mathcal{F}_\sigma$ denote the family of all subsets of proper F_σ subgroups of $\mathbb{T}$.

Theorem Let $\mathcal{D} \subseteq \mathcal{F} \subseteq \mathcal{F}_\sigma$. Then any $\mathcal{F}$-additive set is perfectly meager in transitive sense.

Proof Let $X \neq \emptyset$ be $\mathcal{F}$-additive, P be perfect. Find the sequence $\{n_k\}_{k\in\mathbb{N}}$ as above, and put

$$D_m = \{x : \forall k \geq m \ \ \|n_k x\| \leq 2^{-k}\}.$$

For any m, there is a proper F_σ group F_m containing the set $X + D_m$. Since $\{D_m\}_{m\in\mathbb{N}}$ is increasing, we may assume that also $\{F_m\}_{m\in\mathbb{N}}$ is inceasing, and thus $F = \bigcup F_m$ is a proper F_σ subgroup of $\mathbb{T}$. Denote $E = \{x : \forall^\infty \ \ \|n_k x\| \leq 2^{-k}\}$. Then $F \supseteq \bigcup(X + D_m) = X + E$, and thus F contains X and a translation of E.

Let $y \in \mathbb{T}$ be given. Since F is a proper subgroup, it can be shifted away from itself. Now, since F contains a translation of E, there is also a translation of E in the complement of F. Let $x + E$ be a translation of E contained in $G = \mathbb{T} \setminus (y + F)$. By the construction of $\{n_k\}_{k\in\mathbb{N}}$, $P \cap (x + E)$ is dense in P, and thus also $P \cap G$ is dense in P. Since G is G_δ, $P \cap (y + F)$ is meager in P.