Families of sets and functions related to the uniform convergence of characters

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Notation

$\mathbb{T}$ – the unit circle – compact Polish group

- $\mathbb{T} = \{z \in \mathbb{C}\} : |z| = 1\}$, $x \in \mathbb{R} \mapsto e^{2\pi i x} \in \mathbb{T}$
  multiplication and topology inherited from $\mathbb{C}$
- $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $x \in \mathbb{R} \mapsto \phi(x) = [x]_\sim$ where $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$
  addition modulo integers, quotient topology
  $d(x,y) = \|x - y\|$, where $\|t\| = \min\{|x| : \phi(x) = t\}$, is an invariant metric on $\mathbb{T}$

$C(X,Y)$ – space of all continuous functions $f : X \to Y$ with the topology of uniform convergence

$K(X)$ – space of compact subsets of X (with Vietoris topology),

characters of $\mathbb{T}$ – continuous group homomorphisms $\chi \in C(\mathbb{T},\mathbb{T})$, these are exactly functions $\chi_n(x) = nx$ for $n \in \mathbb{Z}$

Kronecker sets

Definition (Hewitt, Kakutani (1960), Rudin (1962))

A set $E \in K(\mathbb{T})$ is a Kronecker set if

$$(\forall f \in C(\mathbb{T}, \mathbb{T}))(\forall \varepsilon > 0)(\exists n)\ \|\chi_n(x) - f(x)\| < \varepsilon.$$

- We can assume that n is arbitrarily large.
- Every finite independent set is Kronecker, where E is independent if $k_1 x_1 + \cdots + k_n x_n = 0$ implies $k_1 = \cdots = k_n = 0$ for any $k_1, \ldots, k_n \in \mathbb{Z}$, $x_1, \ldots, x_n \in E$.
- Every Kronecker set is independent.
- (Hewitt, Kakutani, 1960) There exists a perfect Kronecker set.
- Every perfect set has a perfect Kronecker subset.
- For any perfect set P there exists a perfect Kronecker set Q such that $P + Q = \mathbb{T}$.

Dirichlet sets

Definition (Kahane (1969))

A set $E \in K(\mathbb{T})$ is a Dirichlet set if
$$(\forall \varepsilon > 0)(\exists n \neq 0)\ \|\chi_n(x)\| < \varepsilon.$$

- We can assume that n is arbitrarily large.
- Every Kronecker set is a Dirichlet set.
- A group generated by a Dirichlet set is proper subgroup of $\mathbb{T}$, hence is meager and of Lebesgue measure zero.
- A shift of a Dirichlet set is a Dirichlet set.

Operations on families of sets and functions

Let $f_k \rightrightarrows f$ denote the uniform convergence of a sequence of functions.

Consider the following binary relation $R \subseteq K(\mathbb{T}) \times C(\mathbb{T}, \mathbb{T})$:
$R(E, f)$ iff there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\chi_{n_k} \rightrightarrows f$ on E.

For $\mathcal{H} \subseteq C(\mathbb{T}, \mathbb{T})$ and $\mathcal{E} \subseteq K(\mathbb{T})$, define
$\mathcal{K}(\mathcal{H}) = \big\{E \in K(\mathbb{T}) : (\forall f \in \mathcal{H})\, R(E, f)\big\}$,
$\mathcal{F}(\mathcal{E}) = \big\{f \in C(\mathbb{T}, \mathbb{T}) : (\forall E \in \mathcal{E})\, R(E, f)\big\}$.

Examples. $\{E : E \text{ is Kronecker}\} = \mathcal{K}(C(\mathbb{T}, \mathbb{T}))$,
$\{E : E \text{ is Dirichlet}\} = \mathcal{K}(\{0\}) = \mathcal{K}(\{\chi_n : n \in \mathbb{Z}\})$.

Galois connection

Pair $(\mathcal{F},\mathcal{K})$ forms a Galois connection between ordered sets $(\mathcal{P}(K(\mathbb{T})),\subseteq)$ and $(\mathcal{P}(C(\mathbb{T},\mathbb{T})),\subseteq)$, i.e., $\mathcal{E}\subseteq\mathcal{F}(\mathcal{H})$ iff $\mathcal{H}\subseteq\mathcal{K}(\mathcal{E})$, for any $\mathcal{E},\mathcal{H}$.

- $\mathcal{K}$ and $\mathcal{F}$ are order-reversing.
- $\mathcal{K}\circ\mathcal{F}$ and $\mathcal{F}\circ\mathcal{K}$ are closure operators.

$(\mathcal{E},\mathcal{H})$ is called a stable pair if $\mathcal{E}=\mathcal{K}(\mathcal{H})$ and $\mathcal{H}=\mathcal{F}(\mathcal{E})$.

- Stable pairs ordered by $(\mathcal{E}_1,\mathcal{H}_1)\leq(\mathcal{E}_2,\mathcal{H}_2)$ iff $\mathcal{H}_1\subseteq\mathcal{H}_2$ iff $\mathcal{E}_1\supseteq\mathcal{E}_2$ form a complete lattice, where $\bigvee_i(\mathcal{E}_i,\mathcal{H}_i)=\Big(\bigcap_i\mathcal{E}_i,\mathcal{F}\big(\bigcap_i\mathcal{E}_i\big)\Big)$, $\bigwedge_i(\mathcal{E}_i,\mathcal{H}_i)=\Big(\mathcal{K}\big(\bigcap_i\mathcal{H}_i\big),\bigcap_i\mathcal{H}_i\Big)$.

Lattice of stable pairs

What is the structure of this lattice?

- top element: $\big(\{E : E \text{ is Kronecker}\}, C(\mathbb{T}, \mathbb{T})\big)$
- bottom element: $(K(\mathbb{T}), \emptyset)$
- one atom: $\big(\{E : E \text{ is Dirichlet}\}, \{\chi_n : n \in \mathbb{Z}\}\big)$

The last statement follows from the following fact.

Lemma

Let $f \in C(\mathbb{T}, \mathbb{T})$. Then $f \in \{\chi_n : n \in \mathbb{Z}\}$ iff $(\forall x, y \in \mathbb{T})\, f(x - y) = f(x) - f(y)$.

Strongly Dirichlet sets

Denote $C = \{f \in C(\mathbb{T}, \mathbb{T}) : f \text{ is constant}\}$,
$TC = \{\chi_n + c : n \in \mathbb{Z} \wedge c \in \mathbb{T}\}$.

Definition

A set $E \in K(\mathbb{T})$ is a strongly Dirichlet set if $E \in \mathcal{K}(C)$.

Fact. $\mathcal{K}(C) = \mathcal{K}(TC)$.

Lemma

Let $f \in C(\mathbb{T}, \mathbb{T})$. Then $f \in TC$ iff
$(\forall x, y \in \mathbb{T})\ f(2x - y) = 2f(x) - f(y)$.

Theorem

$\big(\{E : E \text{ is strongly Dirichlet}\},\ TC\big)$ is a stable pair.

Affinely independent sets

Fact

1. Every strongly Dirichlet set E is affinely independent, i.e., $k_1x_1 + \cdots + k_nx_n = 0$ implies that $k_1 + \cdots + k_n = 0$, for all $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in \mathbb{Z}$.
2. Every finite affinely independent set is strongly Dirichlet.

For $E \subseteq \mathbb{T}$, denote

$$\mathrm{Aff}(E) = \{x \in \mathbb{T} : (\exists x_1, \ldots, x_n \in E)(\exists k, k_1, \ldots, k_n \in \mathbb{Z}) \\ k = k_1 + \cdots + k_n \neq 0 \wedge kx = k_1x_1 + \cdots + k_nx_n\}.$$

Then E is affinely independent iff $E \subseteq \mathrm{Aff}(F)$ for some independent set F.

Theorem

If $E \subseteq \mathbb{T}$ is Kronecker then $\mathrm{Aff}(E)$ is strongly Dirichlet.

m-Dirichlet sets

Let $p \in \mathbb{N}$. Denote $C_m = \{f \in C(\mathbb{T},\mathbb{T}) : (\forall x \in \mathbb{T})\, mf(x) = 0\}$, $TC_m = \{\chi_n : n \in \mathbb{Z}\} + C_m$.

Definition

A set $E \in K(\mathbb{T})$ is an m-Dirichlet set if $E = \mathcal{K}(C_m)$.

Fact. $\mathcal{K}(C_m) = \mathcal{K}(TC_m)$.

Lemma

Let $f \in C(\mathbb{T},\mathbb{T})$. Then $f \in TC_m$ iff $(\forall x, y \in \mathbb{T})\, f(2x - y) = 2f(x) - f(y) \wedge mf(x - y) = mf(x) - mf(y)$.

Theorem

$\big(\{E : E \text{ is } m\text{-Dirichlet}\},\ TC_m\big)$ is a stable pair.

m-independent sets

Definition

Let $m \in \mathbb{N}$. A set $E \subseteq \mathbb{T}$ is m-Dirichlet if $k_1x_1 + \cdots + k_nx_n = 0$ implies that $m \mid k_1 + \cdots + k_n$, for any $x_1, \ldots, x_n \in E$ and $k_1, \ldots, k_n \in \mathbb{Z}$.

Theorem

1. *Every m-Dirichlet set is m-independent.*
2. *Every finite m-independent is m-Dirichlet.*

References

1. Hewitt E., Kakutani S., *A class of multiplicative linear functionals on a measure algebra of a locally compact abelian group*, Illinois J. Math. **4** (1960), 553–574.
2. Rudin W., *Fourier Analysis on Groups*, Interscience Tracts in Pure and Applied Mathematics **12**, Interscience Publishers, 1962.
4. Kahane J.-P., *Approximation par des exponentielles imaginaires; ensembles de Dirichlet et ensembles de Kronecker*, In: Abstract Spaces and Approximation (Proc. Conf., Oberwolfach, 1968), Birkhäuser, 1969, pp. 190–202.
5. Körner T.W., *Some results on Kronecker, Dirichlet and Helson sets*, Ann. Inst. Fourier **20** (1970), 219–234.
6. Körner T.W., *Some results on Kronecker, Dirichlet and Helson sets II*, J. Anal. Math. **27** (1974), 260–388.