Permitted sets and analytic subgroups of $\mathbb{T}$

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Let $(\mathbb{T}, +)$ be the circle group $(\mathbb{R}/\mathbb{Z})$.
For $x \in \mathbb{T}$, let $\|x\|$ denote the distance of x to 0.

Definition Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{T})$. A set $A \subseteq \mathbb{T}$ is $\mathcal{F}$-*permitted* if for every $B \in \mathcal{F}$, the sumset $A + B$ can be covered by some $C \in \mathcal{F}$.

Example $X \subseteq \mathbb{T}$ is an *N-set* if there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of non-negative reals such that $\sum a_n = \infty$ and $\sum a_n \|nx\| < \infty$ for all $x \in X$.

Let $\mathcal{N}$ be the family of all N-sets. $\mathcal{N}$ has a base consisting of F_σ subgroups of $\mathbb{T}$. Every N-set is meager and has Lebesgue measure zero.

Theorem (Arbault, Erdös, 1952) Every countable set is $\mathcal{N}$-permitted.

Problem Does there exist a perfect $\mathcal{N}$-permitted set?

Answers
1. Arbault (1952) – yes
2. Bari (1961) – found error in Arbault's proof
3. Lafontaine (1969) – no; proof seems to be incorrect, no references found
4. Bukovský, Recław, Repický, Kholshchevnikova, Bartoszyński, Scheepers, . . . (1990's) – consistent examples of uncountable $\mathcal{N}$-permitted sets (e.g., γ-set)
5. conjecture (Bukovský) – every $\mathcal{N}$-permitted set is perfectly meager, i.e., meager relatively to any perfect set

Example $X \subseteq \mathbb{T}$ is an *Arbault set* if there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $\|n_k x\| \to 0$ on X. Let $\mathcal{A}$ denote the family of all Arbault sets.

$\mathcal{A}$ has a base consisting of $F_{\sigma\delta}$ subgroups of $\mathbb{T}$. Every Arbault set is meager and has Lebesgue measure zero. $\mathcal{A} \nsubseteq \mathcal{N}$, $\mathcal{N} \nsubseteq \mathcal{A}$.

Theorem A (P.E., 2003) Every $\mathcal{A}$-permitted set is perfectly meager.

Corollary 1. There is no perfect $\mathcal{A}$-permitted set.
2. It is relatively consistent that there is no $\mathcal{A}$-permitted set of the size $\mathfrak{c}$.

Two proofs of Theorem A

1. using a combinatorial characterization of the inclusion in the family $\mathcal{A}$
2. using a strengthening of a theorem by Erdös, Kunen, and Mauldin – simpler and more general

Both proofs make use of Kronecker's theorem.

Characterization of the inclusion in $\mathcal{A}$

Definition Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers.
The subgroup of $\mathbb{T}$ characterized by $\{n_k\}_{k\in\mathbb{N}}$ is the set $A_{\{n_k\}_k} = \{x \in \mathbb{T} : \|n_k\, x\| \to 0\}$.

Theorem (P.E. 2003) Let $\{n_k\}_{k\in\mathbb{N}}$, $\{m_j\}_{j\in\mathbb{N}}$ be increasing sequences of natural numbers, and let $\frac{n_k}{n_{k+1}} \to 0$. Then $A_{\{n_k\}_k} \subseteq A_{\{m_j\}_j}$ iff
there exists a matrix $z \in \mathbb{Z}^{\mathbb{N}\times\mathbb{N}}$ such that
1. $(\forall j)\, m_j = \sum_k z_{k,j}\, n_k$,
2. $(\forall k)\, (\forall^\infty j)\;\; z_{k,j} = 0$,
3. $(\exists c)\, (\forall j)\; \sum_k \left| z_{k,j} \right| < c$.

Remark Condition $\frac{n_k}{n_{k+1}} \to 0$ ensures that the set $A_{\{n_k\}_k}$ has a perfect subset.

Problem Can this condition be omitted?

Erdös–Kunen–Mauldin Theorem

Theorem (Erdös, Kunen, Mauldin 1981) For any perfect set $P \subseteq \mathbb{R}$ there exists a perfect set Q having Lebesgue measure zero such that $P + Q = \mathbb{R}$.

Definition $X \subseteq \mathbb{T}$ is a *Dirichlet set* if there exists an increasing sequence of natural numbers $\{n_k\}_{k \in \mathbb{N}}$ such that $\|n_k x\| \rightrightarrows 0$ on X. Let $\mathcal{D}$ be the family of all Dirichlet sets.

$\mathcal{D}$ has a base consisting of perfect subsets of $\mathbb{T}$. Every Dirichlet set is meager and has Lebesgue measure zero. $\mathcal{D} \subseteq \mathcal{N} \cap \mathcal{A}$.

Theorem (P.E. 2005) For any perfect set $P \subseteq \mathbb{T}$ there exists a Dirichlet set D such that $P + D = \mathbb{T}$.

Corollary If $\mathcal{F} \supseteq \mathcal{D}$ then there is no perfect $\mathcal{F}$-permitted set.

Perfectly meager sets

Definition A set X is *perfectly meager* if for every perfect set P, X is meager relatively to P, i.e., the set $X \cap P$ is meager in the relative topology of P.

Other variants of perfectly meager sets

1. (Zakrzewski) X is *universally meager* iff for any countable family $\mathcal{C}$ of perfect sets, there is an F_σ-set $F \supseteq X$ such that F is meager relatively to every $P \in \mathcal{C}$.
2. (Nowik, Weiss) X is *perfectly meager in transitive sense* iff for any perfect set P there is an F_σ-set $F \supseteq X$ such that F is meager relatively to any translation of P.

perfectly meager in transitive sense $\Rightarrow$ universally meager $\Rightarrow$ perfectly meager

Lemma For any perfect set $P \subseteq \mathbb{T}$ there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that for any sequence $\{y_k\}_{k\in\mathbb{N}}$ in $\mathbb{T}$, the set

$$\left\{x \in \mathbb{T} : (\forall^\infty k)\ \|n_k x - y_k\| \le 2^{-k}\right\} \tag{1}$$

is dense in P.

Theorem B If $\mathcal{F} \subseteq \mathcal{P}(\mathbb{T})$ contains all sets of the form (1) and for every $A \in \mathcal{F}$ there is an F_σ-set $F \supseteq A$ such that $A+F \neq \mathbb{T}$, then every $\mathcal{F}$-permitted set is perfectly meager in transitive sense.

Remark The conditions of Theorem B can be easily checked for $\mathcal{F} = \mathcal{N},\ \mathcal{A}$.

Families generated by analytic subgroups of $\mathbb{T}$

Question What families $\mathcal{F}$ do satisfy the conditions of Theorem B? If $\mathcal{F}$ has a base consisting of subgroups of $\mathbb{T}$, does (*) already follow?

(*) for any $A \in \mathcal{F}$ there is F_σ-set $F \supseteq A$ such that $A + F \neq \mathbb{T}$

Theorem (Laczkovich 1998) Every proper analytic subgroup of $\mathbb{R}$ can be covered by an F_σ-set of Lebesgue measure zero.

Theorem C (P.E. 2006) For every proper analytic subgroup G of $\mathbb{T}$ there exists an F_σ-set $F \supseteq A$ such that $A + F$ has Lebesgue measure zero.

Corollary Let $\mathcal{F}$ has a base consisting of proper analytic subgroups of $\mathbb{T}$ and let $\mathcal{F} \supseteq \mathcal{D}$. Then every $\mathcal{F}$-permitted set is perfectly meager in transitive sense.

References

1. Arbault J., *Sur l'ensemble do convergence absolue d'une série trigonométriqe*, Bull. Soc. Math. France (1952)
2. Erdös P., Kunen K., Mauldin R.D., *Some additive properties of sets of real numbers*, Fund. Math. (1981)
3. Eliaš P., *The inclusions between Arbault sets*, Acta Univ. Carolin. Math. Phys. (2003)
4. Eliaš P., *Arbault permitted sets are perfectly meager*, Tatra Mt. Math. Publ. (2005)
5. Laczkovich M., *Analytic subgroups of the reals*, Proc. Amer. Math. Soc. (1998)
6. Lafontaine J., *Réunions d'ensembles de convergence absolue*, Mém. Soc. Math. France (1969)
7. Nowik A., Weiss T., *Not every Q-set is perfectly meager in the transitive sense*, Proc. Amer. Math. Soc. (2000)