Variations on Kronecker and Dirichlet sets on the circle

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Notation

$\mathbb{T}$ – the unit circle – Polish topological group

multiplicative notation: $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$x \in \mathbb{R} \mapsto e^{2\pi i x} \in \mathbb{T}$

multiplication and topology inherited from $\mathbb{C}$

additive notation: $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$x \in \mathbb{R} \mapsto \phi(x) = [x]_\sim$ where $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$

addition modulo integers, quotient topology

$\|t\| = \min\{|x| : \phi(x) = t\}$, $\|\cdot\| : \mathbb{T} \to \left[0, \frac{1}{2}\right]$

$\varrho(x, y) = \|x - y\|$ is a metric on $\mathbb{T}$

For X, Y metric spaces, $C(X, Y)$ is space of all continuous functions $f : X \to Y$ with the topology of uniform convergence.

Characters of $\mathbb{T}$, i.e., group homomorphisms $\chi \in C(\mathbb{T}, \mathbb{T})$, are exactly functions $\chi_n(x) = nx$ for $n \in \mathbb{Z}$.

Dirichlet's and Kronecker's Theorems

Dirichlet's Theorem

Let $x_1, \dots, x_k \in \mathbb{T}$, $\varepsilon > 0$, $m \in \mathbb{N}$.
There exists $n > m$ such that $\|nx_i\| < \varepsilon$ for $i = 1, \dots, k$.

Kronecker's Theorem

Let $x_1, \dots, x_k \in \mathbb{T}$ are independent, i.e., $\ell_1 x_1 + \cdots + \ell_k x_k = 0$ implies $\ell_1 = \cdots = \ell_k = 0$, for all $\ell_1, \dots, \ell_k \in \mathbb{Z}$.
Let $y_1, \dots, y_k \in \mathbb{T}$, $\varepsilon > 0$, $m \in \mathbb{N}$.
Then there exists $n > m$ such that $\|nx_i - y_i\| < \varepsilon$ for $i = 1, \dots, k$.

Kronecker sets

Definition (Hewitt, Kakutani (1960), Rudin (1962))

A closed set $E \subseteq \mathbb{T}$ is a **Kronecker set** if

$$\forall_{f \in C(\mathbb{T},\mathbb{T})} \forall_{\varepsilon > 0} \forall_m \exists_{n > m} \forall_{x \in E} \ \|nx - f(x)\| < \varepsilon.$$

- Every finite independent set is Kronecker.
- Every Kronecker set is independent.
- (Hewitt, Kakutani, 1960) There exists a perfect Kronecker set.
- (Kaufman, 1967) If P is a perfect totally disconnected set then $\{f \in C(P,\mathbb{T}) : f[P]$ is Kronecker$\}$ is comeager in $C(P,\mathbb{T})$.

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Dirichlet sets

Definition (Kahane (1969))

A closed set $E \subseteq \mathbb{T}$ is a **Dirichlet set** if

$$\forall_{\varepsilon>0} \forall_m \exists_{n>m} \forall_{x\in E} \; \|nx\| < \varepsilon.$$

- Every Kronecker set is Dirichlet.
- Every Dirichlet set has Lebesgue measure zero.
- A shift of a Dirichlet set (i.e., $a + E$ where $a \in \mathbb{T}$ and E is Dirichlet) is a Dirichlet set.
- If E is a Dirichlet set then $E + E$ is a Dirichlet set.

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Kronecker and Dirichlet sets

For $E \subseteq \mathbb{T}$, denote $\mathrm{UA}(E)$ the set of all functions that are uniformly approximable by characters on E, i.e.,

$$\mathrm{UA}(E) = \{f \in C(E,\mathbb{T}) : \forall_{\varepsilon>0} \forall_m \exists_{|n|>m} \forall_{x \in E} \ \|nx - f(x)\| < \varepsilon\}$$

$$= \text{limit points of } \{\chi_n \restriction E : n \in \mathbb{Z}\} \text{ in } C(E,\mathbb{T}).$$

Then E is a Kronecker set iff $\mathrm{UA}(E) = C(E,\mathbb{T})$,
E is a Dirichlet set iff $\mathbf{0}_E \in \mathrm{UA}(E)$.

- (Körner, 1970) There exists a countable independent Dirichlet set which is not Kronecker.
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$E \subseteq \mathbb{T}$ is a Dirichlet set iff

$$\forall_{a\in\mathbb{T}}\,\forall_{\varepsilon>0}\,\forall_m\,\exists_{n>m}\,\forall_{x\in E}\ \|n(x+a)\| < \varepsilon.$$

Can we omit the parentheses?

Definition

A closed set $E \subseteq \mathbb{T}$ is a **strong Dirichlet set** if

$$\forall_{a\in\mathbb{T}}\,\forall_{\varepsilon>0}\,\forall_m\,\exists_{n>m}\,\forall_{x\in E}\ \|nx+a\| < \varepsilon.$$

Then E is a strong Dirichlet set iff $\mathrm{UA}(E)$ contains all constant functions.

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Affinely independent sets

Fact

1. Every strong Dirichlet set E is affinely independent, i.e., $\ell_1 x_1 + \dots \ell_k x_k = 0$ implies $\ell_1 + \dots + \ell_k = 0$, for all $x_1, \dots, x_k \in E$ and $\ell_1, \dots, \ell_k \in \mathbb{Z}$.
2. Every finite affinely independent set is strong Dirichlet.

For $E \subseteq \mathbb{T}$, denote

$$\mathrm{Aff}(E) = \{x \in \mathbb{T} : (\exists x_1, \dots, x_k \in E)(\exists \ell, \ell_1, \dots, \ell_k \in \mathbb{Z})$$
$$n = \ell_1 + \dots + \ell_k \neq 0 \ \wedge \ \ell x = n_1 x_1 + \dots + \ell_k x_k\}.$$

Then E is affinely independent iff $E \subseteq \mathrm{Aff}(F)$ for some independent set F.

Theorem

If $E \subseteq \mathbb{T}$ is Kronecker then $\mathrm{Aff}(E)$ is strong Dirichlet.

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If $E \subseteq \mathbb{T}$ is Kronecker then $\mathrm{Aff}(E)$ is strong Dirichlet.

Kronecker vs. strong Dirichlet vs. Dirichlet sets

Theorem

1. *There exists a countable independent Dirichlet set which is not a strong Dirichlet set.*
2. *There exists a countable independent strong Dirichlet set which is not a Kronecker set.*

Properties of $\mathrm{UA}(E)$

Let $E \subseteq \mathbb{T}$.

- $\mathrm{UA}(E)$ is a closed subgroup of $C(E, \mathbb{T})$.
- $\mathrm{UA}(E) = C(E, \mathbb{T})$ iff E is a Kronecker set.
- $\mathrm{UA}(E) \neq \emptyset$ iff E is a Dirichlet set.
- If $\mathrm{UA}(E) \neq \emptyset$ then $\{\chi_n \restriction E : n \in \mathbb{Z}\} \subseteq \mathrm{UA}(E)$.

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Stable families of functions

For $\mathcal{F} \subseteq C(\mathbb{T}, \mathbb{T})$, denote $K(\mathcal{F}) = \{E \subseteq \mathbb{T} : \mathcal{F} \restriction E \subseteq \mathrm{UA}(E)\}$, where $\mathcal{F} \restriction E = \{f \restriction E : f \in \mathcal{F}\}$.

For $\mathcal{E} \subseteq \mathcal{P}(\mathbb{T})$, denote $\mathcal{K}(\mathcal{E}) = \bigcap_{E \in \mathcal{E}} \mathrm{UA}_{\mathbb{T}}(E)$, where $\mathrm{UA}_{\mathbb{T}}(E) = \{f \in C(\mathbb{T}, \mathbb{T}) : f \restriction E \in \mathrm{UA}(E)\}$.

- $\mathcal{K}(\mathcal{E})$ is a closed subgroup of $C(\mathbb{T}, \mathbb{T})$, for every $\mathcal{E} \subseteq \mathcal{P}(\mathbb{T})$.
- $\mathcal{F} \subseteq \mathcal{K}(K(\mathcal{F}))$, for every $\mathcal{F} \subseteq C(\mathbb{T}, \mathbb{T})$.

Definition

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Stable families of functions

Problem

Characterize stable families $\mathcal{F} \subseteq C(\mathbb{T}, \mathbb{T})$ and the corresponding families $K(\mathcal{F}) \subseteq \mathcal{P}(\mathbb{T})$.

Theorem

Family $\{\chi_n : n \in \mathbb{Z}\} = \mathcal{K}(\mathcal{D})$ is stable, where $\mathcal{D} = \{E \subseteq \mathbb{T} : E$ is a Dirichlet set$\}$.

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References

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