Thin sets of reals related to trigonometric series

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Outline

1. Families of trigonometric thin sets
2. Inclusions between Arbault sets
3. Permitted sets
4. Problem of perfect permitted sets
5. Erdös-Kunen-Mauldin theorem
6. Laczkovich's theorem

Trigonometric series

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$$\sum_{n=0}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx) \tag{1}$$

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Actually, for every nice function f there exists a trigonometric series (1) such that the equality

$$f(x) = \sum_{n=0}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx)$$

holds true for all points x with some exceptions.

Absolute convergence of trigonometric series

Theorem (A. Denjoy, N. Luzin, 1912)

If a trigonometric series (1) *absolutely converges on a set of positive Lebesgue measure then* $\sum_{n=0}^{\infty}(|a_n|+|b_n|)<\infty$ *and hence the series absolutely converges everywhere.*

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N. Luzin proved also analogous theorem for the category.

There exist sets that are both meager and of Lebesgue measure zero for which the conclusion of above theorem holds true, e.g., the standard Cantor set.

Sets of absolute convergence

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A set $X \subseteq \mathbb{T}$ is called a set of absolute convergence (also N-set) if there exists a trigonometric series which absolutely converges on X but is not absolutely converging everywhere.

- N-sets are meager and of Lebesgue measure zero.
- Every countable set is an N-set. There exists a perfect, hence uncountable, N-set. Cantor set is not an N-set.
- A subgroup of $\mathbb{T}$ generated by an N-set is an N-set.
- Every N-set is included in an F_σ N-set.
- Linear transformation of an N-set is again an N-set.

Sets of absolute convergence

For $x \in \mathbb{R}$ denote $\|x\| = \min\{|x - k| : k \in \mathbb{Z}\}$.
We have $\|-x\| = \|x\|$ and $\|x\| - \|y\| \leq \|x + y\| \leq \|x\| + \|y\|$.
Function $\varrho(x, y) = \|x - y\|$ is a metric on $\mathbb{T}$.

We have $\|x\| \leq |\sin \pi x| \leq \pi \|x\|$.

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Theorem (R. Salem, 1941)

A set $X \subseteq \mathbb{T}$ is an N-set if and only if there exist a sequence $\{a_n\}_{n=1}^{\infty}$ of non-negative reals such that $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n \|nx\| < \infty$ for $x \in X$.

The proof is based on the use of Dirichlet theorem.

Families of trigonometric thin sets

A set $X \subseteq \mathbb{T}$ is called

- an N$_0$-set if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \|n_k x\| < \infty$ for $x \in X$,
- a Dirichlet set (also D-set) if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\|n_k x\| \rightrightarrows 0$ on X,
- a pseudo-Dirichlet set (also pD-set) if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $(\forall x \in X)\ (\exists K)\ (\forall k \geq K)\ \ \|n_k x\| < 2^{-k}$,
- an Arbault set (also A-set) if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ such that $\lim_{k \to \infty} \|n_k x\| = 0$ for all $x \in X$.

We denote by $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$ denote families of all N-sets, N$_0$-sets, D-sets, pD-sets, and A-sets, respectively.

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We denote by $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$ denote families of all N-sets, N$_0$-sets, D-sets, pD-sets, and A-sets, respectively.

- $\mathcal{D} \subset p\mathcal{D} \subset \mathcal{N}_0 \subset \mathcal{N}$, $\mathcal{N}_0 \subset \mathcal{A}$, and all inclusions are proper.
- All families except $\mathcal{D}$ are generated by proper Borel subgroups of $\mathbb{T}$.
- There exist two D-sets X and Y such that the group generated by $X \cup Y$ is $\mathbb{T}$.

Relations between thin sets

Theorem (P. Eliaš, 2003)

1. *If* $\sum_{n=0}^{\infty} \frac{n_k}{n_{k+1}} < \infty$ *then* $\left\{x : \sum_{k=0}^{\infty} \|n_k x\| < \infty\right\} \in \mathcal{N}_0 \setminus p\mathcal{D}$.

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2. *If* $\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = 0$ *then* $\left\{x : \lim_{k\to\infty} \|n_k x\| \to 0\right\} \in \mathcal{A} \setminus \mathcal{N}$.

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3. *If* $\lim_{k\to\infty} a_k = 0$, $\sum_{k=0}^{\infty} a_k = \infty$, *and* $\sum_{k=0}^{\infty} a_k \frac{n_k}{n_{k+1}} < \infty$, *then* $\left\{x : \sum_{k=0}^{\infty} a_k \|n_k x\| < \infty\right\} \in \mathcal{N} \setminus \mathcal{A}$.

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Proof goes by a suitable construction of a nested sequence of intervals of the length $1/n_k$.

E.g., every interval I with the length $1/n_k$ has a subinterval J with the length $1/n_{k+1}$ such that for all $x \in J$, $\|n_k x\| \leq n_k/n_{k+1}$.

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Or, if $n_k \leq m \leq n_{k+1}$ and $n_k/n_{k+1} \leq 1/4$ then every interval I with the length $1/n_k$ contains a subinterval J with the length $1/n_{k+1}$ such that for all $x \in J$, $\|mx\| \geq 1/8$.

Inclusions between Arbault sets

Notation. For a given increasing sequence of natural numbers $a = \{a_n\}_{n\in\mathbb{N}}$ denote $A(a) = \left\{x : \lim_{n\to\infty} \|a_n x\| = 0\right\}$.

Problem (D. Maharam, A. Stone)

Characterize those sequences $a = \{a_n\}_{n\in\mathbb{N}}$ for which the set $A(a)$ is uncountable.

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Problem (D. Maharam, A. Stone)

Characterize those sequences $a = \{a_n\}_{n\in\mathbb{N}}$ for which the set $A(a)$ is uncountable.

Question. When does the inclusion $A(a) \subseteq A(b)$ hold true?

Inclusions between Arbault sets

Definition

Let $k \in \mathbb{N}$, and let $z = \{z_{m,n}\}_{m,n \in \mathbb{N}}$ be an infinite matrix of integers. We say that z is a k-bounded matrix if

1. $(\forall n)\ (\exists M)\ (\forall m > M)\ z_{m,n} = 0$, and
2. $(\forall m)\ \sum_{n=0}^{\infty} z_{m,n} \leq k$,

z is a bounded matrix if it is a k-bounded matrix for some $k \in \mathbb{N}$.

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Theorem (P. Eliaš, 2003)

Let $a = \{a_n\}_{n\in\mathbb{N}}$, $b = \{b_m\}_{m\in\mathbb{N}}$ be increasing sequences of natural numbers, and let $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 0$. The following conditions are equivalent.

1. $A(a) \subseteq A(b)$,
2. *there exists a bounded matrix z such that $b =^* z.a$,*
   i.e., $(\exists M)\ (\forall m > M)\ b_m = \sum_{n=0}^{\infty} z_{m,n} a_n$.

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z is a bounded matrix iff

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Sketch of the proof of $\neg 2 \Rightarrow \neg 1$.

Assume $a_0 = 1$. There exists z such that $b = z.a$ and for all m, n,

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- there is n such that the set $M = \{m : z_{m,n} \neq 0\}$ is infinite but $\{m \in M : z_{m,n'} \neq 0\}$ is finite for every $n' > n$, or
- for every n and every infinite set $M \subseteq \{m : z_{m,n} \neq 0\}$ there is $n' > n$ such that the set $\{m \in M : z_{m,n'}\}$ is infinite too.

In each case we find $x \in A(a) \setminus A(b)$. □

Permitted sets

Definition (J. Arbault, 1952)

A set $X \subseteq \mathbb{T}$ is called permitted if for every N-set Y, $X \cup Y$ is an N-set.

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Every countable set is permitted.

Question. Does there exist a perfect permitted set?

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- L. Bukovský conjectured that every permitted set is perfectly meager, i.e., meager relatively to any perfect set

$\mathcal{F}$-permitted sets

Definition

Let $\mathcal{F}$ be a family of set. A set X is called $\mathcal{F}$-permitted if for every $Y \in \mathcal{F}$, $X \cup Y \in \mathcal{F}$. Denote $\mathrm{Perm}(\mathcal{F}) = \{X : X \text{ is } \mathcal{F}\text{-permitted}\}$.

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- If $\mathcal{F}$ is hereditary then $\mathrm{Perm}(\mathcal{F})$ is an ideal.
- If $\mathcal{F}$ is hereditary and has a base consisting of subgroups then X is $\mathcal{F}$-permitted iff it is $\mathcal{F}$-additive, i.e., $X + Y \in \mathcal{F}$ for every $Y \in \mathcal{F}$.

Problem of perfect permitted sets

Theorem (P. Eliaš, 2005)

$\mathcal{A}$-permitted sets are perfectly meager.

Proof uses the characterization of the inclusion between Arbault sets by bounded matrices.

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$\mathcal{F}$-permitted sets are perfectly meager for $\mathcal{F} = p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$.

Proof uses a strengthening of a theorem of P. Erdös, K. Kunen and R. D. Mauldin.

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Theorem (P. Eliaš, 2008)

$\mathcal{F}$-permitted sets are perfectly meager for any hereditary family $\mathcal{F} \supseteq \mathcal{D}$ having a basis consisting of proper analytic subgroups of $\mathbb{T}$.

Proof utilizes a strengthening of a theorem of M. Laczkovich.

A strengthening of Erdös-Kunen-Mauldin theorem

Theorem (P. Erdös, K. Kunen, R. D. Mauldin, 1981)

For every perfect set $P \subseteq \mathbb{R}$ there exists a perfect set M of Lebesgue measure zero such that $P + M = \mathbb{R}$.

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Theorem (P. Eliaš, 2006)

For every perfect set $P \subseteq \mathbb{T}$ there exists a Dirichlet set D such that that $P + D = \mathbb{T}$.

Corollary. If $\mathcal{F}$ is hereditary, $\mathcal{D} \subseteq \mathcal{F}$, and $\mathcal{F}$ has a basis from proper subgroups of $\mathbb{T}$ then there is no perfect $\mathcal{F}$-permitted set.

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Theorem (P. Eliaš, 2006)

For every perfect set $P \subseteq \mathbb{T}$ there exists a pseudo-Dirichlet set D such that $(\forall y \in \mathbb{T})$ $P \cap (D + y)$ is dense in P.

Corollary. If $\mathcal{F}$ is hereditary, $\mathcal{D} \subseteq \mathcal{F}$, and $\mathcal{F}$ has a basis from proper subgroups of $\mathbb{T}$ then every $\mathcal{F}$-permitted set is perfectly meager.

A strengthening of Erdös-Kunen-Mauldin theorem

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Let $x_1, \dots, x_k \in \mathbb{T} \setminus \mathbb{Q}$ *be linearly independent over* $\mathbb{Q}$, $y_1, \dots, y_k \in \mathbb{T}$, $\varepsilon > 0$. *Then there exists arbitrarily large* n *such that* $(\forall i)\ \|nx_i - y_i\| < \varepsilon$.

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Let $x_1, \ldots, x_k \in \mathbb{T} \setminus \mathbb{Q}$ be linearly independent over $\mathbb{Q}$, $y_1, \ldots, y_k \in \mathbb{T}$, $\varepsilon > 0$. Then there exists arbitrarily large n such that $(\forall i)\ \|nx_i - y_i\| < \varepsilon$.

We define by induction a sequence of finite sets $A_k \subseteq P$ linearly independent over $\mathbb{Q}$.

A strengthening of Erdös-Kunen-Mauldin theorem

Prove: for every perfect set P there exists $D \in \mathcal{D}$ such that $P + D = \mathbb{T}$.

Theorem (L. Kronecker)

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Find $\varepsilon_{k+1} \leq \varepsilon_k/2$ such that if $\|x - a\| \leq \varepsilon_{k+1}$ then $\|n_k a - b\| \leq 2^{-k-1}$.
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For a given $y \in \mathbb{T}$, there are $b_k \in B_k$ such that $\|n_k y - b_k\| \leq 2^{-k-2}$.
By induction choose $p_k \in A_k$ such that $\|p_k - p_{k+1}\| \leq \varepsilon_{k+1}/2$.
Sequence $\{p_k\}_{k=0}^{\infty}$ converges to $p \in P$ such that $(\forall k)\ \|n_k(y - p)\| \leq 2^{-k}$, so $y \in P + D$. □

Permitted sets are transitively meager

Definition (A. Nowik, T. Weiss)

A set $X \subseteq \mathbb{T}$ is called transitively meager (or perfectly meager in transitive sense, or AFC')
if for every perfect set P there exists an F_σ-set $F \supseteq P$ such that for every $y \in \mathbb{T}$, $P \cap (F + y)$ is meager in P.

transitively meager $\Rightarrow$ universally meager $\Rightarrow$ perfectly meager

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Corollary. Let $\mathcal{F} \supseteq \mathcal{D}$ be a hereditary family generated by proper subgroups of $\mathbb{T}$, satisfying

$$(\forall E \in \mathcal{F})\ (\exists F_\sigma\text{-set } F \supseteq E)\ E + F \neq \mathbb{T}. \tag{2}$$

Then every $\mathcal{F}$-permitted set is transitively meager.

Fact. Families $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{N}$, $\mathcal{A}$ satisfy condition (2).

A strengthening of Laczkovich's theorem

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Theorem (P. Eliaš)

Let E be a proper analytic subgroup of $\mathbb{T}$. Then there exists an F_σ-set $F \supseteq E$ such that $E + F \in \mathcal{E}$.

Corollary. Every proper analytic subgroup of $\mathbb{T}$ can be separated by an F_σ-set from one of its cosets.

Problem

Is it possible to separate a proper analytic subgroup of $\mathbb{T}$ from any of its cosets?

Analytic sets

Notation.
ω – natural numbers, ω^ω – infinite sequences of natural numbers,
$\omega^{<\omega}$ – finite sequences of natural numbers

A set is called analytic if it is a continuous image of a Borel set in some Polish space.

A set A is analytic iff there exists a Suslin scheme for A, i.e., an indexed system $\{A_t : t \in \omega^{<\omega}\}$ of closed sets such that $s \supseteq t \Rightarrow A_s \subseteq A_t$ and $A = \bigcup_{x\in\omega^\omega} \bigcap_{n\in\omega} A_{x\restriction n}$.

Analytic subgroups

Lemma

Let A be a proper analytic subgroup of $\mathbb{T}$. Then there exists a Suslin scheme $\{A_t : t \in \omega^\omega\}$ for A such that

1. $(\forall t \in \omega^{<\omega} \setminus \emptyset)$ A_t *is nowhere dense*
2. $\sum_{t \in \omega^{<\omega} \setminus \emptyset} \operatorname{diam}(A_t) < \infty$
3. $(\forall t \in \omega^{<\omega})$ $(\exists n)$ $(\forall s \in \omega^\omega, |s| > n)$ $(\exists C$ *countable*$)$ $A_s \subseteq A_t + C$

Theorem (S. Solecki)

Let A be an analytic set, $\mathcal{I}$ be an ideal generated by some family of closed sets. Then either $A \in \mathcal{I}$ or there exists a G_δ-set $G \subseteq A$ such that no portion of G is in $\mathcal{I}$.

a portion of a set means a nonempty relatively open subset

References

1. Arbault J., *Sur l'ensemble de convergence absolue d'une série trigonométrique*, Bull. Soc. Math. France **80** (1952), 253–317.
2. Bukovský L., Kholshchevnikova N. N., Repický M., *Thin sets of harmonic analysis and infinite combinatorics*, Real Anal. Exchange **20** (1994/95), 454–509.
3. Eliaš P., *Arbault permitted sets are perfectly meager*, Tatra Mt. Math Publ. **30** (2005), 135–148.
4. Erdös P., Kunen K., Mauldin R. D., *Some additive properties of sets of real numbers*, Fund. Math. **113** (1981), 187–199.
5. Laczkovich M., *Analytic subgroups of the reals*, Proc. Amer. Math. Soc. **126** (1998), 1783-1790.
6. Solecki S., *Covering analytic sets by families of closed sets*, J. Symbolic Logic **59** (1994), 1022–1031.