A HIERARCHY OF THIN SETS RELATED TO THE BOUNDEDNESS OF TRIGONOMETRIC SERIES

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Abstract. We study the family $\mathcal{B}_0$ of the sets on which some series of the form $\sum_{k\in\mathbb{N}} |\sin \pi n_k x|$ is uniformly bounded. We show that the families $\mathcal{B}_0^c$ of all sets admitting the boundary c form a hierarchy which is incontinuous with respect to the operations of intersection and union.

Let us recall two kinds of thin sets studied in harmonic analysis. A set $X \subseteq [0,1]$ is called an *N-set* (in honour of V. V. Nemytskiĭ) if there exists a trigonometric series absolutely converging on X but not converging absolutely everywhere; it is called an *N_0-set* if there exists a series of the form $\sum_{k\in\mathbb{N}} |\sin \pi n_k x|$ converging on X. J. Arbault [1] showed that the family $\mathcal{N}_0$ of all N_0-sets is a proper subfamily of the family $\mathcal{N}$ of all N-sets, and both families share some common properties.

In the paper [3] we examined several modifications of the definitions of the families $\mathcal{N}$ and $\mathcal{N}_0$, and compared the obtained families one to another. We showed that the family $\mathcal{B}_0$ of all sets on which some series of the form $\sum_{k\in\mathbb{N}} |\sin \pi n_k x|$ is uniformly bounded, differs from previously known "classical" families. In the present paper we continue the study of the family $\mathcal{B}_0$ considering the hierarchy of the families $\mathcal{B}_0^c$ of all sets admitting the boundary c. We show that if $c < d$ then $\mathcal{B}_0^c$ is a proper subfamily of $\mathcal{B}_0^d$, and $\bigcup_{d<c} \mathcal{B}_0^d \subsetneqq \mathcal{B}_0^c \subsetneqq \bigcap_{d>c} \mathcal{B}_0^d$ for every $c > 0$.

For a review of families of trigonometric thin sets, some historical notes, and also for many new results we refer the reader to the paper [2].

We shall deal with the quotient group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$; however, we will not distinguish between the elements of $\mathbb{T}$ and the reals, or between the functions defined on $\mathbb{T}$ and the corresponding periodic functions on $\mathbb{R}$. For a real x, let $[x]$ denote the integer part of x and let $\|x\|$ denote the distance of x to the nearest integer, i. e. $[x] = \max\{k \in \mathbb{Z} : k \le x\}$ and $\|x\| = \min\{|x-k| : k \in \mathbb{Z}\}$. Let us note that $\|x\| - \|y\| \le \|x+y\| \le \|x\| + \|y\|$ and $\|-x\| = \|x\|$ for all $x, y \in \mathbb{T}$. The space $\mathbb{T}$ equipped with the metric $\varrho(x,y) = \|x-y\|$ is a compact topological group.

In an accordance with [3], we define B_0-sets as follows.

Definition 1. A set $X \subseteq \mathbb{T}$ is a *B_0-set* if there exist an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that the series $\sum_{k\in\mathbb{N}} \|n_k x\|$ is uniformly bounded on X, i. e. there exists a real c such that $\sum_{k\in\mathbb{N}} \|n_k x\| \le c$ for all $x \in X$. The family of all B_0-sets is denoted by $\mathcal{B}_0$.

Since $2\|x\| \le |\sin \pi x| \le \pi \|x\|$ for all $x \in \mathbb{T}$, it is clear that the previous definition remains equivalent if we replace $\sum_{k\in\mathbb{N}} \|n_k x\|$ by $\sum_{k\in\mathbb{N}} |\sin \pi n_k x|$.

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Proposition 2. *A set $X \subseteq \mathbb{T}$ is a B_0-set if and only if there exist an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers and a sequence $\{a_k\}_{k\in\mathbb{N}}$ of elements of $\mathbb{T}$ such that the series $\sum_{k\in\mathbb{N}} \|n_k x + a_k\|$ is uniformly bounded on X.*

Proof. It follows immediately from Lemma 3 below. □

Lemma 3. *Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathbb{T}$. Then for every $\varepsilon > 0$ there exists an increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ of natural numbers such that $\sum_{j\in\mathbb{N}} \|m_j x\| \leq \sum_{k\in\mathbb{N}} \|n_k x + a_k\| + \varepsilon$ for every $x \in \mathbb{T}$.*

Proof. We use a classical argument. Since $\mathbb{T}$ is compact, there exists an increasing function $h : \mathbb{N} \to \mathbb{N}$ such that the sequence $\left\{n_{h(2j+1)} - n_{h(2j)}\right\}_{j\in\mathbb{N}}$ is increasing and $\sum_{j\in\mathbb{N}} \left\|a_{h(2j+1)} - a_{h(2j)}\right\| \leq \varepsilon$. It suffices to put $m_j = n_{h(2j+1)} - n_{h(2j)}$. □

However, we do not know the answer for the following question.

Problem 4. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathbb{T}$. Does there exist a sequence $\{m_j\}_{j\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} \|m_j x\| \leq \sum_{k\in\mathbb{N}} \|n_k x + a_k\|$ for all $x \in \mathbb{T}$?

Let us now define the families $\mathcal{B}_0^c$. Here the use of the series $\sum_{k\in\mathbb{N}} \|n_k x\|$ and $\sum_{k\in\mathbb{N}} |\sin \pi n_k x|$ is not equivalent; for simplicity we will consider the first one.

Definition 5. Let c be a positive real. A set $X \subseteq \mathbb{T}$ is called a *B_0^c-set* if there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\sum_{k\in\mathbb{N}} \|n_k x\| \leq c$ for all $x \in X$. The family of all B_0^c-sets is denoted by $\mathcal{B}_0^c$.

We show that in the previous definition, '$\leq$' can be replaced by '$<$'.

Lemma 6. *Let $c > 0$, $X \subseteq \mathbb{T}$, and let $\{n_k\}_{k\in\mathbb{N}}$ be such that $\sum_{k\in\mathbb{N}} \|n_k x\| \leq c$ for all $x \in X$. Then there exists $j \in \mathbb{N}$ such that $\sum_{k>j} \|n_k x\| < c$ for all $x \in X$.*

Proof. Suppose for a contradiction that for every j there exists $x_j \in X$ such that $\sum_{k>j} \|n_k x_j\| \geq c$. Thus $\|n_k x_j\| = 0$ for all $k \leq j$ and $\left\|n_{k_j} x_j\right\| > 0$ for some $k_j > j$. Hence for every j and k such that $k \geq k_j$ we have $\left\|n_{k_j} x_j\right\| > 0$ and $\left\|n_{k_j} x_k\right\| = 0$, and thus $x_j \neq x_k$. It follows that the set $\{x_j : j \in \mathbb{N}\}$ is infinite. However, this is impossible since for every k there are only finitely many $x \in \mathbb{T}$ such that $\|n_k x\| = 0$. □

Let us recall that a set $X \subseteq \mathbb{T}$ is called a *Dirichlet set* (a *D-set*) if there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that the sequence $\{\|n_k x\|\}_{k\in\mathbb{N}}$ is uniformly converging to 0 on X. The family of all D-sets is denoted by $\mathcal{D}$. From the definition we can immediately see that $\bigcup_{c>0} \mathcal{B}_0^c = \mathcal{B}_0$ and $\bigcap_{c>0} \mathcal{B}_0^c = \mathcal{D}$.

We do not know whether an analogue of Proposition 2 holds true in the case of B_0^c-sets.

Problem 7. Let $c > 0$, let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of elements of $\mathbb{T}$. Does there exist a sequence $\{m_j\}_{j\in\mathbb{N}}$ such that for all $x \in \mathbb{T}$, $\sum_{j\in\mathbb{N}} \|m_j x\| \leq c$ whenever $\sum_{k\in\mathbb{N}} \|n_k x + a_k\| \leq c$?

Fix a positive real c. Let us denote by $\mathcal{B}_0^{*c}$ the family of all sets $X \subseteq \mathbb{T}$ for which there exist sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{a_k\}_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} \|n_k x + a_k\| \leq c$ for all

$x \in X$. Clearly $\mathcal{B}_0^c \subseteq \mathcal{B}_0^{*c}$, and Problem 7 asks whether the equality holds true. ¿From Lemma 3 it follows that $\mathcal{B}_0^{*c} \subseteq \bigcap_{d>c} \mathcal{B}_0^d$. Thus for every $c > 0$ we have

$$\bigcup_{d<c} \mathcal{B}_0^d = \bigcup_{d<c} \mathcal{B}_0^{*d} \subseteq \mathcal{B}_0^c \subseteq \mathcal{B}_0^{*c} \subseteq \bigcap_{d>c} \mathcal{B}_0^d = \bigcap_{d>c} \mathcal{B}_0^{*d}.$$

In Propositions 10 and 11 we will show that the first and the last inclusions are proper.

Let $\{m_j\}_{j\in\mathbb{N}}$ be an increasing sequence of natural numbers and let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of elements of $\mathbb{T}$. For every positive integer n, let $\alpha(n), \beta(n) \in \mathbb{N}$ be such that $n = 2^{\alpha(n)}\beta(n)$ and $\beta(n)$ is odd.

Lemma 8. *Let $p, r \in \mathbb{N}$, $p \geq 1$. Assume that the set $\{\alpha(m_j) : j \in \mathbb{N}\}$ is infinite. There exist $j, q \in \mathbb{N}$ and $s \in \{-1, 1\}$ such that $q \geq r$ and $\left\| s m_j 2^{-q2^p} + a_j \right\| \geq 2^{-2^p}$.*

Proof. Put $J = \left\{ j \in \mathbb{N} : \|a_j\| \geq 2^{-2^p} \right\}$. Clearly either $\{\alpha(m_j) : j \in J\}$ or $\{\alpha(m_j) : j \notin J\}$ is infinite.

In the first case find $j \in J$ such that $\alpha(m_j) \geq r2^p$, and put $q = r$ and $s = 1$. Then $s m_j 2^{-q2^p}$ is an integer and thus $\left\| s m_j 2^{-q2^p} + a_j \right\| = \|a_j\| \geq 2^{-2^p}$.

In the latter case find $j \notin J$ such that $\alpha(m_j) \geq (r-1)2^p$ and put $q = [\alpha(m_j)2^{-p}] + 1$. Then clearly $q \geq r$. We also have $0 < q2^p - \alpha(m_j) \leq 2^p$ and thus $\left\| m_j 2^{-q2^p} \right\| = \left\| \beta(m_j) 2^{\alpha(m_j) - q2^p} \right\| \geq 2^{-2^p}$. Since $\|a_j\| \leq 2^{-2^p} \leq 2^{-2}$, we can conclude that either $\left\| m_j 2^{-q2^p} + a_j \right\| \geq 2^{-2^p}$ or $\left\| m_j 2^{-q2^p} - a_j \right\| \geq 2^{-2^p}$. $\square$

Lemma 9. *Let θ be a real such that $0 < \theta \leq 1/2$, let $\{p_k\}_{k\in\mathbb{N}}$ be a sequence of positive integers, and let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive reals. Assume that $\lim_{j\to\infty} \alpha(m_j) = \infty$. There exists $x \in \mathbb{T}$ such that*

1. *$\|x\| \leq \theta$;*
2. *there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\left\| 2^{2^{n_k}} x \right\| \leq 2^{-2^{p_k}} + 2^{-2^{n_k}}$ for all $k \in \mathbb{N}$, and $\left\| 2^{2^n} x \right\| \leq 2^{-2^n}$ for all $n \notin \{n_k : k \in \mathbb{N}\}$;*
3. *there exists an increasing sequence $\{j_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\|m_{j_k} x + a_{j_k}\| \geq 2^{-2^{p_k}} - \varepsilon_k$ for all $k \in \mathbb{N}$.*

Proof. We will define sequences $\{n_k\}_{k\in\mathbb{N}}$, $\{j_k\}_{k\in\mathbb{N}}$, $\{q_k\}_{k\in\mathbb{N}}$, and $\{s_k\}_{k\in\mathbb{N}}$ by induction. Fix $k \in \mathbb{N}$ and suppose that n_i, j_i, q_i, s_i are defined for all $i < k$. Find $r \geq 2$ such that

(a) $2^{-r2^{p_k}} \leq 2^{-k-1}\theta$;
(b) $r2^{p_k} > 2^{n_{k-1}+1}$ if $k \geq 1$;
(c) $2^{-r2^{p_k}} \leq 2^{i-k} 2^{-2^{n_i+1}}$ for all $i < k$;
(d) $2^{-r2^{p_k}} \leq 2^{i-k} \varepsilon_i / m_{j_i}$ for all $i < k$.

Let $l \in \mathbb{N}$ be such that $l > j_{k-1}$ if $k \geq 1$, and $\alpha(m_j) \geq q_{k-1} 2^{p_{k-1}}$ for all $j \geq l$. By Lemma 8 there exist $j_k \geq l$, $q_k \geq r$, and $s_k \in \{-1, 1\}$ such that

(e) $\left\| m_{j_k} s_k 2^{-q_k 2^{-p_k}} + a_{j_k} \right\| \geq 2^{-2^{p_k}}$.

Put $n_k = \max\{i : 2^i < q_k 2^{p_k}\}$.

Let $x = \sum_{k\in\mathbb{N}} s_k 2^{-q_k 2^{p_k}}$. By (a), $|x| \leq \sum_{k\in\mathbb{N}} 2^{-q_k 2^{p_k}} \leq \sum_{k\in\mathbb{N}} 2^{-k-1}\theta = \theta$, and thus (1) holds.

To show (2), fix $k \in \mathbb{N}$. If $k \geq 1$ then by (b), $q_k 2^{p_k} > 2^{n_{k-1}+1}$. Thus $n_k > n_{k-1}$ and hence $q_i 2^{p_i} < 2^{n_k}$ for all $i < k$. Since $q_k \geq 2$, we have $2^{n_k+1} \geq q_k 2^{p_k} \geq 2^{p_k+1}$

and thus $n_k \geq p_k$. Hence 2^{n_k} is a multiple of 2^{p_k} and we have $q_k 2^{p_k} \geq 2^{n_k} + 2^{p_k}$. Using (c) we obtain that

$$\text{(f)} \quad \begin{aligned} \sum_{i \geq k} 2^{-q_i 2^{p_i}} &\leq 2^{-q_k 2^{p_k}} + \sum_{i>k} 2^{k-i} 2^{-2^{n_k+1}} \\ &\leq 2^{-2^{n_k}-2^{p_k}} + 2^{-2^{n_k+1}} = 2^{-2^{n_k}} \left(2^{-2^{p_k}} + 2^{-2^{n_k}}\right). \end{aligned}$$

Hence $\left\|2^{2^{n_k}} x\right\| \leq 2^{2^{n_k}} \sum_{i \geq k} 2^{-q_i 2^{p_i}} \leq 2^{-2^{p_k}} + 2^{-2^{n_k}}$, and thus the first part of (2) holds. To show the latter one, let $n \notin \{n_k : k \in \mathbb{N}\}$ and put $k = \min\{i : n_i > n\}$. Then for all $i < k$ we have $n > n_i$ and thus $q_i 2^{p_i} \leq 2^n$. Hence by (f),

$$\left\|2^{2^n} x\right\| \leq 2^{2^n} \sum_{i \geq k} 2^{-q_i 2^{p_i}} \leq 2^{2^n - 2^{n_k}} \left(2^{-2^{p_k}} + 2^{-2^{n_k}}\right) \leq 2^{2^n - 2^{n+1}} = 2^{-2^n}.$$

To show (3), fix again $k \in \mathbb{N}$. From the choice of l it follows that $j_k > j_{k-1}$ if $k \geq 1$, and $\alpha(m_{j_k}) \geq q_i 2^{p_i}$ for all $i < k$. Hence by (e) and (d),

$$\begin{aligned} \|m_{j_k} x + a_{j_k}\| &\geq \left\|m_{j_k} s_k 2^{-q_k 2^{p_k}} + a_{j_k}\right\| - m_{j_k} \sum_{i>k} 2^{-q_i 2^{p_i}} \\ &\geq 2^{-2^{p_k}} - m_{j_k} \sum_{i>k} 2^{k-i} \varepsilon_k / m_{j_k} = 2^{-2^{p_k}} - \varepsilon_k. \end{aligned}$$

□

For $n \in \mathbb{N}$, denote $\phi(n) = \sum_{k \geq n} 2^{-2^k}$, $\psi(0) = 1$, and $\psi(n) = 2^{-2^n} \sum_{k<n} 2^{2^k}$ if $n \geq 1$. It can be easily shown that both $\{\phi(n)\}_{n\in\mathbb{N}}$ and $\{\psi(n)\}_{n\in\mathbb{N}}$ are decreasing and converging to 0.

Proposition 10. *For every $c > 0$ there exists $X \subseteq \mathbb{T}$ such that*

(1) *there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\sum_{k\in\mathbb{N}} \|n_k x\| \leq c$ for all $x \in X$;*
(2) *for every increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ of natural numbers, every sequence $\{a_j\}_{j\in\mathbb{N}}$ of elements of $\mathbb{T}$, and for every $\eta > 0$ there exists $x \in X$ such that $\sum_{j\in\mathbb{N}} \|m_j x + a_j\| > c - \eta$.*

Proof. Put $X = \left\{x \in \mathbb{T} : \sum_{n\in\mathbb{N}} : \left\|2^{2^n} x\right\| \leq c\right\}$. Clearly X satisfies the condition (1).

To show (2), fix $\eta > 0$ and sequences $\{m_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$. We will find $x \in X$ such that $\sum_{j\in\mathbb{N}} \|m_j x + a_j\| > c - \eta$. We will consider two cases.

(a) If $\lim_{j\to\infty} \alpha(m_j) = \infty$ then we can find $N \in \mathbb{N}$, a sequence $\{p_k\}_{k\in\mathbb{N}}$ of positive integers and a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive reals such that $\sum_{k\in\mathbb{N}} 2^{-2^{p_k}} - \sum_{k\in\mathbb{N}} \varepsilon_k > c - \eta$ and $\sum_{k\in\mathbb{N}} 2^{-2^{p_k}} + \phi(N) + \psi(N) \leq c$. Put $\theta = 2^{-2^N}$. There exists $x \in \mathbb{T}$ satisfying the conditions (1)–(3) of Lemma 9. We have

$$\sum_{n\in\mathbb{N}} \left\|2^{2^n} x\right\| = \sum_{n<N} \left\|2^{2^n} x\right\| + \sum_{n\geq N} \left\|2^{2^n} x\right\| \leq \sum_{n<N} 2^{2^n} \theta + \sum_{k\in\mathbb{N}} 2^{-2^{p_k}} + \sum_{n\geq N} 2^{-2^n} \leq c$$

and

$$\sum_{j\in\mathbb{N}} \|m_j x + a_j\| \geq \sum_{k\in\mathbb{N}} \|m_{j_k} x + a_{j_k}\| \geq \sum_{k\in\mathbb{N}} \left(2^{-2^{p_k}} - \varepsilon_k\right) > c - \eta.$$

(b) If the sequence $\{\alpha(m_j)\}_{j\in\mathbb{N}}$ does not tend to infinity then we will find $x \in X$ such that the sequence $\{\|m_j x + a_j\|\}_{j\in\mathbb{N}}$ does not tend to 0. We can suppose that $\lim_{j\to\infty} \|a_j\| = 0$ (otherwise take $x = 0$). There exist $M \in \mathbb{N}$ and an increasing sequence $\{j_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\alpha(m_{j_k}) = M$ for all k. Find $N \in \mathbb{N}$ such that $2^N > M$ and $\psi(N) \le c$. For $x = 2^{-2^N}$ we get

$$\sum_{n\in\mathbb{N}} \left\|2^{2^n} x\right\| = \sum_{n<N} 2^{2^n - 2^N} = \psi(N) \le c.$$

Since $\|m_{j_k}x + a_{j_k}\| \ge \|m_{j_k}x\| - \|a_{j_k}\| \ge 2^{M-2^N} - \|a_{j_k}\|$ for all $k \in \mathbb{N}$, the sequence $\{\|m_j x + a_j\|\}_{j\in\mathbb{N}}$ does not converge to 0. □

Proposition 11. *For every $c > 0$ there exists $X \subseteq \mathbb{T}$ such that*

(1) *for every $\eta > 0$ there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} \|n_k x\| < c + \eta$ for all $x \in X$;*
(2) *for every increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ of natural numbers and for every sequence $\{a_j\}_{j\in\mathbb{N}}$ of elements of $\mathbb{T}$ there exists $x \in X$ such that $\sum_{j\in\mathbb{N}} \|m_j x + a_j\| \ge c$.*

Proof. Put

$$X = \left\{x \in \mathbb{T} : \textstyle\sum_{n\ge N} \left\|2^{2^n} x\right\| \le c + \phi(N) + \psi(N) \text{ for all } N \in \mathbb{N}\right\}.$$

Clearly X satisfies the condition (1).

To show (2), fix sequences $\{m_j\}_{j\in\mathbb{N}}$, $\{a_j\}_{j\in\mathbb{N}}$. We will find $x \in X$ such that $\sum_{j\in\mathbb{N}} \|m_j x + a_j\| \ge c$. Again let us discuss two cases.

(a) Assume that $\lim_{j\to\infty} \alpha(m_j) = \infty$. Let us choose a sequence $\{p_k\}_{k\in\mathbb{N}}$ of positive integers such that $\sum_{k\in\mathbb{N}} 2^{-2^{p_k}} = c$. Find $t \in \mathbb{N}$ such that $\left\|m_0 2^{-2^t} + a_0\right\| > 0$ and put $\theta = \left\|m_0 2^{-2^t} + a_0\right\|/2$. Fix a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive reals such that $\sum_{k\in\mathbb{N}} \varepsilon_k = \theta$ and find $l \in \mathbb{N}$ such that $\alpha(m_j) \ge 2^t$ for all $j \ge l$. By Lemma 9 there exists $x' \in \mathbb{T}$ such that

(1') $\|x'\| \le \theta/m_0$;
(2') there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\left\|2^{2^{n_k}} x'\right\| \le 2^{-2^{p_k}} + 2^{-2^{n_k}}$ for all $k \in \mathbb{N}$, and $\left\|2^{2^n} x'\right\| \le 2^{-2^n}$ for all $n \notin \{n_k : k \in \mathbb{N}\}$;
(3') there exists an increasing sequence $\{j_k\}_{k\in\mathbb{N}}$ of natural numbers such that $j_k \ge l$ and $\|m_{j_k} x' + a_{j_k}\| \ge 2^{-2^{p_k}} - \varepsilon_k$ for all $k \in \mathbb{N}$.

Put $x = x' + 2^{-2^t}$. Then for every $N \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{n\ge N} \left\|2^{2^n} x\right\| &\le \sum_{n\ge N} \left\|2^{2^n} x'\right\| + \sum_{n\ge N} \left\|2^{2^n - 2^t}\right\| \\ &\le \sum_{k\in\mathbb{N}} 2^{-2^{p_k}} + \sum_{n\ge N} 2^{-2^n} + \sum_{N\le n<t} 2^{2^n - 2^t} \le c + \phi(N) + \psi(N),\end{aligned}$$

since the last sum equals 0 if $t \le N$ and is not greater than $\psi(t) < \psi(N)$ if $t > N$. Moreover,

$$\begin{aligned}\sum_{j\in\mathbb{N}} \|m_j x + a_j\| &\ge \|m_0 x + a_0\| + \sum_{k\in\mathbb{N}} \|m_{j_k} x + a_{j_k}\| \\ &\ge \left\|m_0 2^{-2^t} + a_0\right\| - \|m_0 x'\| + \sum_{k\in\mathbb{N}} \|m_{j_k} x' + a_{j_k}\| \\ &\ge \theta + \sum_{k\in\mathbb{N}} 2^{-2^{p_k}} - \sum_{k\in\mathbb{N}} \varepsilon_k = c.\end{aligned}$$

(b) If the sequence $\{\alpha(m_j)\}_{j\in\mathbb{N}}$ does not tend to infinity then similarly as in the proof of Proposition 10 there exists $x \in \mathbb{T}$ such that $\sum_{n\in\mathbb{N}} \left\|2^{2^n} x\right\| \le c$ and the sequence $\{\|m_j x + a_j\|\}_{j\in\mathbb{N}}$ does not tend to 0. □

The following theorem summarizes our results concerning the hierarchy of the families $\mathcal{B}_0^c$ and $\mathcal{B}_0^{*c}$.

Theorem 12. *Let c be a positive real. Then*

$$\bigcup_{d<c} \mathcal{B}_0^d = \bigcup_{d<c} \mathcal{B}_0^{*d} \subsetneqq \mathcal{B}_0^c \subseteq \mathcal{B}_0^{*c} \subsetneqq \bigcap_{d>c} \mathcal{B}_0^d = \bigcap_{d>c} \mathcal{B}_0^{*d}.$$

Corollary 13. *If $c < d$ then $\mathcal{B}_0^c \subsetneqq \mathcal{B}_0^d$ and $\mathcal{B}_0^{*c} \subsetneqq \mathcal{B}_0^{*d}$.*

Let us note that the proofs of Propositions 10 and 11 can be easily adopted for showing the following statements.

Proposition 14. *For every $c > 0$ there exists $X \subseteq \mathbb{T}$ such that*

(1) *there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ of natural numbers such that $\sum_{k\in\mathbb{N}} |\sin \pi n_k x| \le c$ for all $x \in X$;*
(2) *for every increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ of natural numbers, every sequence $\{a_j\}_{j\in\mathbb{N}}$ of elements of $\mathbb{T}$, and for every $\eta > 0$ there exists $x \in X$ such that $\sum_{j\in\mathbb{N}} |\sin \pi(m_j x + a_j)| > c - \eta$.*

Proposition 15. *For every $c > 0$ there exists $X \subseteq \mathbb{T}$ such that*

(1) *for every $\eta > 0$ there exists an increasing sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} |\sin \pi n_k x| < c + \eta$ for all $x \in X$;*
(2) *for every increasing sequence $\{m_j\}_{j\in\mathbb{N}}$ of natural numbers and for every sequence $\{a_j\}_{j\in\mathbb{N}}$ of elements of $\mathbb{T}$ there exists $x \in X$ such that $\sum_{j\in\mathbb{N}} |\sin \pi(m_j x + a_j)| \ge c$.*

Hence it is possible to prove results analogous to Theorem 12 and Corollary 13, concerning the families of sets obtained by replacing the expression $\sum_{k\in\mathbb{N}} \|n_k x\|$ in Definition 5 by $\sum_{k\in\mathbb{N}} |\sin \pi n_k x|$.

References

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