SOME INEQUALITIES INVOLVING INTEGRAL MEANS

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Abstract. A class of generalized weighted quasi-arithmetic means in the integral form $M_{[a,b],g}(p,f)$ is studied using the weighted integral form of Jensen's inequality. In particular, various inequalities and properties of the generalized weighted quasi-arithmetic means are established with respect to the properties of input functions p, f and g. Some well known inequalities as a consequence of our results are derived.

Introduction

In the discrete case, a mean of a nonnegative n-tuple of real numbers $a = (a_1, \dots, a_n)$ with respect to a weight vector $p = (p_1, \dots, p_n)$ of positive real numbers, where $\sum_{i=1}^n p_i = 1$, $n \in \mathbb{N}$ (the set of all positive integers), is defined with the formula

$$M(p,a) = \varphi^{-1}\left(\sum_{i=1}^n p_i \varphi(a_i)\right), \tag{1}$$

where φ is a continuous and strictly monotone function which has its inverse function φ^{-1} satisfying the same conditions, [6]. Here φ is called the Kolmogoroff-Nagumo function associated with (1). The mean of the form (1) is also referred to as the *quasi-arithmetic mean*. This class of means comprises the commonly used algebraic means and also other types of aggregation operators.

A considerable amount of literature about the concept of mean (or average) and the properties of several means (e.g. the arithmetic mean, the geometric mean, the power mean, the harmonic mean) has been appeared in the 19th century. A. N. Kolmogoroff [10] and M. Nagumo [13] were the first who investigated the characteristic properties of means in general. They considered mostly the case of equal weights. The generalization to arbitrary weights and the characterization of means of the form (1) are due to B. de Finetti [3], B. Jessen [8], T. Kitagawa [9], J. Aczél [1] and many others.

Integral analogue of means (1) was established in 1930ties by A. N.Kolmogoroff, M. Nagumo and B. de Finetti. They showed that all types of the so called *intrinsic*

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means may be expressed via

$$\mathcal{M}_F(\varphi) = \varphi^{-1}\left(\int_{\mathbb{R}} \varphi(x)\, dF(x)\right), \tag{2}$$

where $F(x)$ is a distribution and $\varphi(x)$ is a continuous real increasing function on $\mathbb{R}$ (the set of all real numbers), cf. [6]. The integral (2) is meant in the sense of Lebesgue-Stieltjes. This form of means coincides with class of the so called *integral φ-means*. Recently, it is well known that many types of means may be rewritten according to the pattern given with the formula (2), e.g. the Stolarsky's means [19].

In this paper we study the quasi-arithmetic non-symmetrical weighted mean (3) proposed by F. Qi in [18]. In Section 2, we state an analogue of Jensen's inequality for the weighted integral means as well as its conversion. This enables us to derive various inequalities for means $M_{[a,b],g}(p, f)$, $a < b$, with respect to the convexity property of functions f and g. Also a comparison theorem between generalized weighted quasi-arithmetic means is established. As a consequence we derive the weighted integral version of Wang-Wang's geometric-harmonic mean inequality, Chebyshev's integral inequality and the well known Pólya-Knopp's inequality.

1. Preliminaries

Let $[a, b] \subset \mathbb{R}$, $a < b$, be an interval. Denote by $L_1([a, b])$ the vector space of all real Lebesgue integrable functions defined on the interval $[a, b]$ with the classical Lebesgue measure. Let us denote by $L_1^+([a, b])$ the positive cone of $L_1([a, b])$, i.e. the vector space of all real positive Lebesgue integrable functions on $[a, b]$. In what follows $\|p\|_{[a,b]}$ denotes the finite L_1-norm of a function $p \in L_1^+([a, b])$.

Definition 1.1. Let $(p, f) \in L_1^+([a, b]) \times L_1^+([a, b])$ and $g : [0, \infty] \to \mathbb{R}$ be a real continuous and strictly monotone function. The *generalized weighted quasi-arithmetic mean of function f with respect to weight function p* is a number $M_{[a,b],g}(p, f) \in \mathbb{R}$, where

$$M_{[a,b],g}(p, f) = g^{-1}\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x)g(f(x))\, dx\right), \tag{3}$$

where g^{-1} denotes the inverse function to the function g.

In what follows, g is always a real continuous and strictly monotone function (in accordance with Definition 1.1).

Means $M_{[a,b],g}(p, f)$ include many commonly used two variable integral means as particular cases when taking the suitable functions p, f and g. For instance,

(a) for $g(x) = x = I(x)$ (the identity function) we get the *weighted arithmetic mean*

$$M_{[a,b],g}(p, f) = A_{[a,b]}(p, f) = \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x)f(x)\, dx;$$

(b) for $g(x) = x^{-1}$ we obtain the *weighted harmonic mean*

$$M_{[a,b],g}(p,f) = H_{[a,b]}(p,f) = \left(\frac{1}{\|p\|_{[a,b]}} \int_a^b \frac{p(x)}{f(x)}\, dx \right)^{-1};$$

(c) for $g(x) = x^r$ we have the *weighted power mean of order r*

$$M_{[a,b],g}(p,f) = M^{[r]}(f;p;a,b) = \begin{cases} \left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) f(x)^r\, dx \right)^{\frac{1}{r}}, & r \neq 0 \\ \exp\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) \ln f(x)\, dx \right), & r = 0 \end{cases}.$$

The case $r = 0$ corresponds to the *weighted geometric mean* and $r = 2$ is called the *weighted Euclidean quadratic mean.*

(d) if we replace $p(x)$ by $p(x) f^r(x)$ and $g(x)$ by x^{s-r} in (3), then we get the *generalized weighted mean values*

$$M_{[a,b],g}(p,f) = M_{p,f}(r,s;a,b) = \left(\frac{\int_a^b p(x) f^s(x)\, dx}{\int_a^b p(x) f^r(x)\, dx} \right)^{\frac{1}{s-r}};$$

(e) if $p(x)$ is a constant function on $[0, 2\pi]$ and $f(x)$ is of the form

$$f(x) = \begin{cases} (u^n \cos^2 x + v^n \sin^2 x)^{\frac{1}{n}}, & n \neq 0 \\ u^{\cos^2 x} v^{\sin^2 x}, & n = 0 \end{cases},$$

then we obtain the *generalization of arithmetic-geometric mean of Gauss*

$$M_{[a,b],g}(p,f) = M_{g,n}(u,v) = g^{-1} \left(\frac{1}{2\pi} \int_0^{2\pi} g(f(x))\, dx \right)$$

for $n \neq 0$. Choosing $n = 2$ and $g(x) = x^{-1}$ we get original Gauss' arithmetic-geometric mean. For $n = -2$ and $g(t) = t^{-2}$ the mean may be found in [16], the case $n = 1$ and $g(t) = \ln t$ was studied in [2].

Note that means $M_{[a,b],g}(p,f)$ generalize also logarithmic means $L(a,b)$, identric means $I(a,b)$, one-parameter means $J_r(a,b)$, abstracted means $M_g(a,b)$, extended mean values $E(r,s;a,b)$, generalized logarithmic means $S_r(a,b)$, and many others. Hence, from $M_{[a,b],g}(p,f)$ we may deduce most of the two variable means. Further possible extension of means $M_{[a,b],g}(p,f)$ could be considered when f is of the form $f(\theta) = |h(re^{\imath\theta})|$, where $0 < r < 1$ and h is an analytic function in the open unit disk $D = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. In that case choosing $a = 0$, $b = 2\pi$, $g(x) = x^q$ for $0 < q < \infty$ and $p(x) \equiv 1$ on $[0, 2\pi]$, we get the integral mean of order q, cf. [4],

$$M_{[a,b],g}(p,f) = M_q(r,h) = \left(\frac{1}{2\pi} \int_0^{2\pi} |h(re^{\imath\theta})|^q\, d\theta \right)^{\frac{1}{q}}.$$

2. Jensen's inequality and its conversion

Many mathematical investigations deal with problems about operator of means depending of the behavior of the input functions p, f and g, resp. how functions behave under the action of means. The most known case is that of Jensen's convex functions, which originally deal with the arithmetic mean.

In general measure theoretical notation the Jensen's inequality theorem sounds as follows: let (Ω, A, μ) be a measurable space, such that $\mu(\Omega) = 1$. If f is a real μ-integrable function and φ is a convex (concave) function on the range of f, then

$$\varphi\left(\int_\Omega f\ d\mu\right) \leq \int_\Omega \varphi \circ f\ d\mu,$$

$$\left(\varphi\left(\int_\Omega f\ d\mu\right) \geq \int_\Omega \varphi \circ f\ d\mu,\right)$$

cf. [17]. The following Lemma 2.1 is a direct application of this general statement to our particular situation when $\mu([a, b]) = 1$ and $d\mu = \frac{p(x)}{\|p\|_{[a,b]}} dx$.

Lemma 2.1 (Jensen's Inequality)**.** *Let $(p, f) \in L_1^+([a, b]) \times L_1^+([a, b])$ such that $\alpha < f(x) < \beta$ for all $x \in [a, b]$, where $-\infty < \alpha < \beta < \infty$.*

(i) *If g is a convex function on (α, β), then*

$$g\big(A_{[a,b]}(p, f)\big) \leq A_{[a,b]}(p, g \circ f).$$

(ii) *If g is a concave function on (α, β), then*

$$\Big(g\big(A_{[a,b]}(p, f)\big) \geq A_{[a,b]}(p, g \circ f)\Big),$$

where $A_{[a,b]}(p, f)$ denotes the weighted arithmetic mean of the function f on $[a, b]$.

Corollary 2.2. *Let $(p, f) \in L_1^+([a, b]) \times L_1^+([a, b])$ such that $\alpha < f(x) < \beta$ for all $x \in [a, b]$, where $-\infty < \alpha < \beta < \infty$.*

(i) *If g is a convex increasing or concave decreasing function on (α, β), then*

$$A_{[a,b]}(p, f) \leq M_{[a,b],g}(p, f).$$

(ii) *If g is a convex decreasing of concave increasing function on (α, β), then*

$$A_{[a,b]}(p, f) \geq M_{[a,b],g}(p, f).$$

Proof. Let g be a convex increasing function. According to Jensen's inequality we get

$$g^{-1}\left(g\big(A_{[a,b]}(p, f)\big)\right) \leq g^{-1}\left(A_{[a,b]}(p, g \circ f)\right),$$

which is equivalent to

$$A_{[a,b]}(p, f) \leq M_{[a,b],g}(p, f).$$

Proofs of the remaining parts are similar. □

Some elementary properties of $M_{[a,b],g}(p,f)$ derived by the use of the weighted integral analogue of Jensen's inequality may be found in [5]. Also some well known inequalities among weighted means in integral form may be obtained as its direct corollaries.

The following theorem corresponds to some conversions of the Jensen's inequality for convex (concave) functions in the case of $M_{[a,b],g}(p,f)$.

Theorem 2.3. *Let $(p,f)\in L_1^+([a,b])\times L_1^+([a,b])$, such that $f:[a,b]\to[\alpha,\beta]$, and $g:[\alpha,\beta]\to\mathbb{R}$, where $-\infty<\alpha<\beta<\infty$.*

(i) *If g is convex on $[\alpha,\beta]$, then*

$$g\Big(M_{[a,b],g}(p,f)\Big)\le\frac{g(\alpha)\Big(\beta-A_{[a,b]}(p,f)\Big)}{\beta-\alpha}+\frac{g(\beta)\Big(A_{[a,b]}(p,f)-\alpha\Big)}{\beta-\alpha}.$$

(ii) *If g is concave on $[\alpha,\beta]$, then*

$$g\Big(M_{[a,b],g}(p,f)\Big)\ge\frac{g(\alpha)\Big(\beta-A_{[a,b]}(p,f)\Big)}{\beta-\alpha}+\frac{g(\beta)\Big(A_{[a,b]}(p,f)-\alpha\Big)}{\beta-\alpha}.$$

Proof. We will prove only the item (i). The item (ii) may be proved analogously.

Suppose that g is a convex function on the interval $[\alpha,\beta]$. Let us consider the following integral

$$\int_a^b p(x)g(f(x))\,dx.$$

Putting

$$\lambda(x)=\frac{f(x)-\alpha}{\beta-\alpha},\tag{4}$$

we have

$$f(x)=(1-\lambda(x))\alpha+\lambda(x)\beta,$$

for all $x\in[a,b]$. Since $\alpha\le f(x)\le\beta$ for all $x\in[a,b]$ and g is convex on $[\alpha,\beta]$, we have

$$\begin{aligned}\int_a^b p(x)g(f(x))\,dx &\le \int_a^b p(x)\Big((1-\lambda(x))g(\alpha)+\lambda(x)g(\beta)\Big)\,dx\\ &= g(\alpha)\int_a^b p(x)\Big(1-\lambda(x)\Big)\,dx+g(\beta)\int_a^b p(x)\lambda(x)\,dx.\end{aligned}$$

By (4) we get

$$\int_a^b p(x)\lambda(x)\,dx=\frac{1}{\beta-\alpha}\left(\int_a^b p(x)f(x)\,dx-\alpha\,\|p\|_{[a,b]}\right)$$

and therefore

$$\begin{aligned}\int_a^b p(x)g(f(x))\,dx &\le \frac{g(\alpha)}{\beta-\alpha}\left(\beta\,\|p\|_{[a,b]}-\int_a^b p(x)f(x)\,dx\right)\\ &+\frac{g(\beta)}{\beta-\alpha}\left(\int_a^b p(x)f(x)\,dx-\alpha\,\|p\|_{[a,b]}\right).\end{aligned}$$

Since $\|p\|_{[a,b]}$ is positive and finite, we may write

$$g\Big(M_{[a,b],g}(p,f)\Big) = \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x)g(f(x))\,dx$$
$$\leq \frac{g(\alpha)\left(\beta - \frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)\,dx\right)}{\beta-\alpha} + \frac{g(\beta)\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)\,dx - \alpha\right)}{\beta-\alpha}$$
$$= \frac{g(\alpha)\left(\beta - A_{[a,b]}(p,f)\right)}{\beta-\alpha} + \frac{g(\beta)\left(A_{[a,b]}(p,f)-\alpha\right)}{\beta-\alpha}.$$

The proof is complete. $\square$

3. Some inequalities among means

In this section we investigate some more properties of the class of generalized weighted quasi-arithmetic means expressed in the integral form. Summarizing elementary properties of $M_{[a,b],g}(p,f)$ related to convexity (concavity) of functions f, g we obtain the following easy lemma. For the proof, cf. [5].

Lemma 3.1. *Let $(p,k) \in L_1^+([a,b]) \times L_1^+([a,b])$ and let $h_i \in L_1^+([a,b])$ be a sequence of functions, $i = 1,\dots,n$; $n \in \mathbb{N}$. Let $\delta \in \mathbb{R}$.*

(a) *If $g : [0,\infty) \to \mathbb{R}$ is a convex function and $f : [0,\infty) \to \mathbb{R}$ is a concave function on $[a,b]$, then*
  (i) *$M_{[a,b],g}(p,k) \geq M_{[a,b],(-g)}(p,k)$ and $M_{[a,b],f}(p,k) \leq M_{[a,b],(-f)}(p,k)$;*
  (ii) *$M_{[a,b],f}(p,k) \leq M_{[a,b],g}(p,k)$;*
  (iii) *$M_{[a,b],f}(p,\delta) \leq \delta \leq M_{[a,b],g}(p,\delta)$, where $\delta = \delta(x) \geq 0$ is a constant function for $x \in [a,b]$;*
  (iv) *$\sum_{i=1}^n M_{[a,b],f}(p,h_i) \leq M_{[a,b],g}\left(p, \sum_{i=1}^n h_i\right)$;*
  (v) *if $f(x) \geq g(x)$ for all $x \in [a,b]$, then $M_{[a,b],f}(p,g) \leq A_{[a,b]}(p,g) \leq A_{[a,b]}(p,f) \leq M_{[a,b],g}(p,f)$.*

(b) *If $g : [0,\infty) \to \mathbb{R}$ is a concave function and $f : [0,\infty) \to \mathbb{R}$ is a convex function on $[a,b]$, then the above inequalities (i)–(v) are in the reversed order.*

Jensen's inequality provides very important tool for a kind of comparison of means. Therefore, we state the following comparison theorem between means $M_{[a,b],g_k}(p,f)$ and $M_{[a,b],g_{k-1}}(p,f)$.

Theorem 3.2. *Let $(p,f) \in L_1^+([a,b]) \times L_1^+([a,b])$ such that $f : [a,b] \to [\alpha,\beta]$. Let g_k, $k = 1,2,\dots,m$ be one to one functions defined on $[\alpha,\beta]$. Let us denote $G_1 = g_1$, $G_2 = g_2 \circ g_1^{-1}, \dots, G_{k+1} = g_{k+1} \circ g_k^{-1}$ for $k = 1,2,\dots,m-1$.*

(i) *If either G_k are concave increasing or convex decreasing on the range of g_k, for $k = 1,2,\dots,m$, then*

$$M_{[a,b],g_k}(p,f) \leq M_{[a,b],g_{k-1}}(p,f), \quad k = 1,2,\dots,m.$$

(ii) *If either G_k are concave decreasing or convex increasing on the range of g_k, for $k = 1, 2, \dots, m$, then*

$$M_{[a,b],g_k}(p, f) \geq M_{[a,b],g_{k-1}}(p, f), \quad k = 1, 2, \dots, m.$$

Proof. Let us suppose that $G_k = g_k \circ g_{k-1}^{-1}$ are concave increasing functions for $k = 1, 2, \dots, m$. By Jensen's inequality we have

$$\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) G_k\Big(g_{k-1}(f(x))\Big)\, dx \leq G_k\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g_{k-1}(f(x))\, dx\right),$$

for $k = 1, 2, \dots, m$. Since the functions G_k are increasing, it follows that g_k are increasing too, and therefore for $k = 1, 2, \dots, m$ we get

$$\begin{aligned} & g_k^{-1}\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) G_k\Big(g_{k-1}(f(x))\Big)\, dx\right) \\ & \leq (g_k^{-1} \circ G_k)\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g_{k-1}(f(x))\, dx\right). \end{aligned}$$

Using the fact $g_k^{-1} \circ G_k = g_{k-1}^{-1}$ and $G_k \circ g_{k-1} = g_k$, we finally obtain the inequality

$$g_k^{-1}\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g_k(f(x))\, dx\right) \leq g_{k-1}^{-1}\left(\frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g_{k-1}(f(x))\, dx\right),$$

for $k = 1, 2, \dots, m$, which corresponds to $M_{[a,b],g_k}(p, f) \leq M_{[a,b],g_{k-1}}(p, f)$.

Proofs of the remaining parts are similar. □

Lemma 3.3. *Let $p \in L_1^+([a, b])$ and $f : [a, b] \to \mathbb{R}$ be a continuous and integrable function with the continuous first derivative on (a, b).*

(i) *If f is a strictly monotone and convex function on $[a, b]$, then*

$$A_{[a,b]}(p, f) \leq f\left(A_{[a,b]}(p(x)f'(x), x)\right).$$

(ii) *If f is a strictly monotone and concave function on $[a, b]$, then*

$$A_{[a,b]}(p, f) \geq f\left(A_{[a,b]}(p(x)f'(x), x)\right).$$

Proof. Let us define θ by

$$\theta = \frac{\int_a^b p(x) f'(x) x\, dx}{\int_a^b p(x) f'(x)\, dx}. \tag{5}$$

Since f is strictly monotone on $[a, b]$, it follows that $\theta \in (a, b)$. The convexity of f ensures that f' is nondecreasing on (a, b) and

$$f(x) + f'(x)(\theta - x) \leq f(\theta),$$

cf. [14]. Multiplying both sides of the above inequality by $\frac{p(x)}{\|p\|_{[a,b]}}$, we have

$$\frac{p(x)f(x)}{\|p\|_{[a,b]}} + \frac{p(x)f'(x)(\theta - x)}{\|p\|_{[a,b]}} \leq \frac{p(x)f(\theta)}{\|p\|_{[a,b]}}. \tag{6}$$

Integrating (6) with respect to x we may write

$$\frac{\int_a^b p(x)f(x)\,dx}{\|p\|_{[a,b]}} + \theta \cdot \frac{\int_a^b p(x)f'(x)\,dx}{\|p\|_{[a,b]}} - \frac{\int_a^b p(x)f'(x)x\,dx}{\|p\|_{[a,b]}} \leq f(\theta).$$

Replacing θ by (5) we obtain

$$\frac{\int_a^b p(x)f(x)\,dx}{\|p\|_{[a,b]}} \leq f\left(\frac{\int_a^b p(x)f'(x)x\,dx}{\int_a^b p(x)f'(x)\,dx}\right),$$

i.e. $A_{[a,b]}(p,f) \leq f\left(A_{[a,b]}(p(x)f'(x),x)\right)$. □

Example 3.4. Let $[a,b] = (0,1/2]$ and suppose $p \in L_1^+((0,1/2])$. Let $f(x) = \ln\frac{1-x}{x}$ on $(0,1/2]$. It is easy to verify that function $f(x)$ is strictly decreasing and convex on $(0,1/2]$ and $f'(x) = \frac{1}{x(x-1)}$, i.e. the assumptions of Lemma 3.3(a) are satisfied. Thus,

$$\begin{aligned}\frac{1}{\|p\|_{(0,1/2]}}\int_0^{1/2} p(x)\ln\frac{1-x}{x}\,dx &\leq \ln\left(\frac{\int_0^{1/2}\frac{p(x)}{x(x-1)}\,dx - \int_0^{1/2}\frac{p(x)}{x-1}\,dx}{\int_0^{1/2}\frac{p(x)}{x-1}\,dx}\right)\\ &= \ln\left(\frac{\int_0^{1/2}\frac{p(x)}{x}\,dx}{\int_0^{1/2}\frac{p(x)}{1-x}\,dx}\right).\end{aligned}$$

The above inequality may be rewritten as follows

$$\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{1/2} p(x)\ln\frac{1-x}{x}\,dx\right) \leq \frac{\int_0^{1/2}\frac{p(x)}{x}\,dx}{\int_0^{1/2}\frac{p(x)}{1-x}\,dx},$$

which is equivalent to

$$\frac{\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{1/2} p(x)\ln(1-x)\,dx\right)}{\exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{1/2} p(x)\ln x\,dx\right)} \leq \frac{\int_0^{1/2}\frac{p(x)}{x}\,dx}{\int_0^{1/2}\frac{p(x)}{1-x}\,dx}. \tag{7}$$

Using the notation

$$G_{(0,1/2]}(p(x),g(x)) = \exp\left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{1/2} p(x)\ln g(x)\,dx\right),$$

for the weighted geometric mean and

$$H_{(0,1/2]}(p(x),g(x)) = \left(\frac{1}{\|p\|_{(0,1/2]}}\int_0^{1/2}\frac{p(x)}{g(x)}\,dx\right)^{-1},$$

for the weighted harmonic mean, we may rewrite the inequality (7) as follows

$$\frac{G_{(0,1/2]}(p(x),1-x)}{G_{(0,1/2]}(p(x),x)} \leq \frac{H_{(0,1/2]}(p(x),1-x)}{H_{(0,1/2]}(p(x),x)},$$

which is equivalent to the weighted integral inequality of Wang-Wang, cf. [11], in the form

$$\frac{H_{(0,1/2]}(p(x),x)}{H_{(0,1/2]}(p(x),1-x)} \leq \frac{G_{(0,1/2]}(p(x),x)}{G_{(0,1/2]}(p(x),1-x)}.$$

The following theorem is an easy corollary of Lemma 3.3.

Theorem 3.5. *Let $p \in L_1^+([a,b])$. Suppose $f:[a,b]\to[\alpha,\beta]$ is a continuous and integrable function on $[a,b]$ with the continuous first derivative on (a,b), and $g:[\alpha,\beta]\to\mathbb{R}$.*

(i) *If either g is a convex increasing or concave decreasing function on $[\alpha,\beta]$ and f is a strictly monotone and concave function on $[a,b]$, then*

$$f\left(A_{[a,b]}(p(x)f'(x),x)\right) \leq M_{[a,b],g}(p,f).$$

(ii) *If either g is a concave increasing or convex decreasing function on $[\alpha,\beta]$ and f is a strictly monotone and convex function on $[a,b]$, then*

$$f\left(A_{[a,b]}(p(x)f'(x),x)\right) \geq M_{[a,b],g}(p,f).$$

Proof. Let g be a concave decreasing function on $[\alpha,\beta]$ and f be a strictly monotone and concave function on $[a,b]$. From Corollary 2.2 and Lemma 3.3 we immediately get the inequalities

$$M_{[a,b],g}(p,f) \geq A_{[a,b]}(p(x),f(x)) \geq f\left(A_{[a,b]}(p(x)f'(x),x)\right).$$

Remaining parts may be proved analogously. □

Theorem 3.6. *Let $(p,f)\in(L_1^+([a,b])\times L_1^+([a,b]))$, where $f:[a,b]\to[\alpha,\beta]$ is a continuous function with the continuous first derivative on (a,b). Let $g:[\alpha,\beta]\to\mathbb{R}$.*

(i) *If either g is convex increasing or concave decreasing on $[\alpha,\beta]$, then*

$$A_{[a,b]}(p,f) \leq f(a) + M_{[a,b],g}\left(p(x),\int_a^x f'(t)\,dt\right).$$

(ii) *If either g is concave increasing or convex decreasing on $[\alpha,\beta]$, then*

$$A_{[a,b]}(p,f) \geq f(a) + M_{[a,b],g}\left(p(x),\int_a^x f'(t)\,dt\right).$$

Proof. Consider the case when g is convex increasing on $[\alpha,\beta]$. The direct calculation yields

$$M_{[a,b],g}\left(p(x),\int_a^x f'(t)\,dt\right) \geq A_{[a,b]}\left(p(x),\int_a^x f'(t)\,dt\right)$$
$$= \frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)\,dx - \frac{f(a)}{\|p\|_{[a,b]}}\int_a^b p(x)\,dx = A_{[a,b]}(p,f) - f(a).$$

□

Similar results we may obtain when considering integrals $\int_x^b f'(t)\,dt$. An easy corollary for the generalized weighted quasi-arithmetic mean of product of two functions follows from the Chebyshev's integral inequality in the following form, cf. [12].

Lemma 3.7 (Chebyshev's Inequality)**.** *Let $p \in L_1^+([a,b])$ and let $h,k : [a,b] \to \mathbb{R}$ be two integrable functions, both increasing or both decreasing on $[a,b]$. Then*

$$A_{[a,b]}(p,h) \cdot A_{[a,b]}(p,k) \le A_{[a,b]}(p,hk). \tag{8}$$

If one of the functions h or k is nonincreasing and the other nondecreasing, then the inequality in (8) is reversed.

Theorem 3.8. *Let $p \in L_1^+([a,b])$, let $h,k : [a,b] \to [\alpha,\beta]$ be two integrable functions and $f : [\alpha,\beta] \to \mathbb{R}$. Let g be a real continuous monotone function defined on the range of hk.*

(i) *If g is convex increasing or concave decreasing, f is concave increasing or convex decreasing and h,k are either both increasing or both decreasing functions, then*

$$M_{[a,b],f}(p,h) \cdot M_{[a,b],f}(p,k) \le M_{[a,b],g}(p,hk).$$

(ii) *If g is concave increasing or convex decreasing, f is convex increasing or concave decreasing and one of the functions h,k is nonincreasing and the other one nondecreasing, then*

$$M_{[a,b],f}(p,h) \cdot M_{[a,b],f}(p,k) \ge M_{[a,b],g}(p,hk).$$

Proof. Let us prove the item (i). Suppose that h,k are both increasing functions, g is convex increasing and f is concave increasing function. From Corollary 2.2 it follows that

$$M_{[a,b],f}(p,h) \le A_{[a,b]}(p,h) \quad \text{and} \quad M_{[a,b],f}(p,k) \le A_{[a,b]}(p,k).$$

Since h,k are both increasing functions, then

$$M_{[a,b],f}(p,h) \cdot M_{[a,b],f}(p,k) \le A_{[a,b]}(p,h) \cdot A_{[a,b]}(p,k).$$

Applying Lemma 3.7 and Corollary 2.2 we get

$$M_{[a,b],f}(p,h) \cdot M_{[a,b],f}(p,k) \le A_{[a,b]}(p,hk) \le M_{[a,b],g}(p,hk).$$

Hence the result. Similarly we may prove the remaining parts. □

4. Some applications

In this section we deal with the geometric mean operator $\mathbf{G} : L_1^+([0,\infty)) \to L_1^+([0,\infty))$ defined as follows: If $f \in L_1^+([0,\infty))$, then

$$[\mathbf{G}f](x) = \exp\left(\frac{1}{x}\int_0^x \ln f(t)\,dt\right), \quad x \in (0,\infty), \tag{9}$$

i.e. we consider $g^{-1}(\cdot) = \exp(\cdot)$ and the weighted function $p(t) \equiv 1$.

There exist many inequalities involving the geometric mean operator (9), cf. [15]. We will prove the following generalization of the inequality considered in [7] and give an easy corollary related to Pólya-Knopp's inequality and its weighted form.

Theorem 4.1. *Let n be a natural and r, s, q be real numbers satisfying $2r^n - 1 > q - sr^n$. If*

$$\int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt < \infty,$$

then

$$\begin{aligned}\int_0^\infty x^q \, \exp\left[r^{2n} x^{-r^n} \int_0^x t^{r^n-1} \ln |x^{-s} f(t)|\, dt\right] dx \\ \le \; Ce \int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt, \qquad (10)\end{aligned}$$

where

$$C = \frac{r^n}{2r^n + r^n s - q - 1}.$$

Proof. Using the substitution $t = xy$, we may rewrite the left side of (10) as follows

$$\int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln |x^{-s} f(xy)|\, dy\right] dx.$$

Since $e = \exp\left[-r^{2n} \int_0^1 y^{r^n-1} \ln y\, dy\right]$, we have that (10) has the form

$$\begin{aligned}\int_0^\infty x^q \, \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln |x^{-s} f(xy)|\, dy\right] dx \le \\ \le C \cdot \exp\left[-r^{2n} \int_0^1 y^{r^n-1} \ln y\, dy\right] \cdot \int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt.\end{aligned}$$

The direct calculation yields

$$\begin{aligned}C \int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt &\ge \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln y\, dy\right] \\ &\times \int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln |x^{-s} f(xy)|\, dy\right] dx \\ &= \int_0^\infty x^q \exp\left[r^{2n} \int_0^1 y^{r^n-1} \ln |x^{-s} y f(xy)|\, dy\right] dx.\end{aligned}$$

Therefore, we get the inequality

$$\int_0^\infty x^q \exp\left[r^n \int_0^1 y^{r^n-1} \ln |x^{-s} y f(xy)|^{r^n}\, dy\right] dx \le C \int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt.$$

By Jensen's inequality, the left side is dominated by

$$r^n \int_0^\infty x^q \cdot \left(\int_0^1 y^{r^n-1} |x^{-s} y f(xy)|^{r^n}\, dy\right) dx. \qquad (11)$$

Applying Fubini's theorem to (11) and using the substitution $t = xy$, we get

$$r^n \int_0^1 y^{r^n-1} \cdot \left(\int_0^\infty x^{q-sr^n} |f(xy)|^{r^n}\, dx \right) dy =$$

$$= r^n \int_0^1 y^{2r^n+sr^n-q-2} \cdot \left(\int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt \right) dy =$$

$$= \frac{r^n}{2r^n + sr^n - q - 1} \cdot \int_0^\infty t^{q-sr^n} |f(t)|^{r^n}\, dt.$$

Hence the result. $\square$

Note that from the inequality (10) we may obtain some well known inequalities as direct consequences. For instance, the following Pólya–Knopp's inequality.

Corollary 4.2. *Let $r = 1, q = s = n = 0$ and $f(t) \geq 0$. Then the inequality (10) reduces to*

$$\int_0^\infty \exp\left[\frac{1}{x}\int_0^x \ln f(t)\, dt\right] dx \leq e \int_0^\infty f(x)\, dx, \tag{12}$$

or equivalently,

$$\int_0^\infty [\mathbf{G}f](x)\, dx \leq e \int_0^\infty f(x)\, dx.$$

Some weighted versions of Pólya–Knopp's inequality (12) may be also directly derived from (10), for example:

$$\int_0^\infty x^q \exp\left[\frac{1}{x}\int_0^x \ln f(t)\, dt\right] dx \leq \frac{e}{1-q} \int_0^\infty x^q f(x)\, dx,$$

for every $q < 1$, which is more general than (12).

Using Theorem 4.1 we may obtain some other interesting integral inequalities. One of them is the Cochran–Lee's type inequality.

Corollary 4.3. *Let $s = 0$, let $r^n = \lambda > 0$ and q be real numbers such that $2\lambda - 1 > q$. Then (10) has the form*

$$\int_0^\infty x^q \exp\left[\frac{\lambda^2}{x^\lambda}\int_0^x t^{\lambda-1} \ln |f(t)|\, dt\right] dx \leq Ce \int_0^\infty x^q |f(x)|^\lambda\, dx,$$

where

$$C = \frac{\lambda}{2\lambda - q - 1}.$$

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