On Integrable Functions in Complete Bornological Locally Convex Spaces

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Abstract. A Lebesgue-type integration theory in complete bornological locally convex topological vector spaces was introduced by the first author in [17]. In this paper we continue developing this integration technique and formulate and prove some theorems on integrable functions as well as some convergence theorems. An example of Dobrakov integral in non-metrizable complete bornological locally convex spaces is given.

1 Introduction

We can observe that theories containing a certain compatible collection of basic theorems, a calculus, lie in the focus of the present measure and integration investigations. This calculus makes possible and determines further applications of the integral in a particular branch of mathematics.

In a series of papers I. Dobrakov developed a Lebesgue-type integration theory which deals with the $L(\mathbf{X}, \mathbf{Y})$-valued measure σ-additive in the strong operator topology in the Banach spaces. Among them the papers [9] and [10] are fundamental. It is well-known that the Dobrakov integral yields a greater class of integrable functions than the also well-known (Lebesgue-type) integral of R. G. Bartle, cf. [1], considering the same measure and set systems, cf. [9]. The Dobrakov integral is defined in Banach spaces. There are also some possible generalizations of this integral, e.g. to locally convex spaces, cf. [22], [25], or [4], [5], [6]. A definition of a Bartle-type integral in complete bornological locally convex topological vector spaces in the sense of Hogbe-Nlend [19] may be found in [3].

There is a natural tendency to generalize integrations from Banach spaces to "higher floors". For instance, there is a question how to construct a theory of integration in locally convex spaces which are non-metrizable. The bornological character of the bilinear integration theory developed in [22] shows the fitness of developing a bilinear integration theory in the context of bornological convex vector spaces.

One of the equivalent definitions of the complete bornological locally convex topological vector spaces (C. B. L. C. S., for short) is to define these spaces as the inductive limits of Banach spaces. Therefore there is a natural question whether the Dobrakov integral in C. B. L. C. S. may be defined as a finite sum of Dobrakov integrals in various Banach spaces, the choice of which may depend on the function which we integrate. In [13, 14, 15, 16] we may find an

[1] *Mathematics Subject Classification (2000):* Primary 46G10; Secondary 06F20
Key words and phrases: Inductive limit of Banach spaces, Dobrakov integral, Locally convex spaces, Bornology, Sequential convergence.
Acknowledgement. This paper was supported by Grants VEGA 2/5065/05 and APVT-51-006904.

integration technique which generalizes the Dobrakov integration theory to C. B. L. C. S. and in [17] it is shown that such an integral may be constructed. The sense of this seemingly complicated theory is that, at the present, this is the only integration theory which completely generalizes the Dobrakov integration to a class of non-metrizable locally convex topological vector spaces. A suitable class of operator measures in C. B. L. C. S. which allow such a generalization is a class of all σ-additive bornological measures. For lattices of set functions connected with an operator valued measure in C. B. L. C. S., see [15], and for convergences of measurable functions in C. B. L. C. S., see [14]. The construction and existence of product measures in C. B. L. C. S. in connection with this integration technique is given in [2].

The aim of this paper is to continue developing the integration technique introduced in [17] and investigate some theorems on integrable functions for this integral. Some convergence theorems will also be proved and basic properties of the integral stated. An example of Dobrakov-type integral in C. B. L. C. S. is given.

2 Preliminaries

In order to state our results, we give a brief development of a theory of integration in C. B. L. C. S. and collect the needed definitions and results from [13], [14], [15] and [17].

2.1 Complete bornological locally convex spaces

The description of the theory of C. B. L. C. S. may be found in [19], [20] and [21].

Let $\mathbf{X}, \mathbf{Y}$ be two C. B. L. C. S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}, \mathfrak{B}_{\mathbf{Y}}$, respectively. Recall that a *(separable) Banach disk* in $\mathbf{X}$ is a set which is closed, absolutely convex and the linear span of which is a (separable) Banach space. Let us denote by $\mathcal{U}$ the set of all Banach disks U in $\mathbf{X}$ such that $U \in \mathfrak{B}_{\mathbf{X}}$. So, the space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U$, $U \in \mathcal{U}$,

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U,$$

cf. [20], where $\mathbf{X}_U$ is a $\mathbb{K}$-linear span of $U \in \mathcal{U}$ and the family $\mathcal{U}$ is directed by inclusion and forms the basis of bornology $\mathfrak{B}_{\mathbf{X}}$ (analogously for $\mathbf{Y}$ and $\mathcal{W}$). The basis $\mathcal{U}$ in the inductive limit need not be unambiguous and, in particular, it may be chosen such that $\mathbf{X}_U$, $U \in \mathcal{U}$, are separable.

We say that the basis $\mathcal{U}$ of the bornology $\mathfrak{B}_{\mathbf{X}}$ has the *vacuum vector* [2] $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let the bases $\mathcal{U}$, $\mathcal{W}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, with marked elements $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, respectively.

Since $\mathbf{X}_U$, $U \in \mathcal{U}$, in the definition of C. B. L. C. S. is a Banach space, it is enough to deal with sequences instead of nets and therefore we introduce the following bornological convergence in the sense of Mackey. Let $\mathcal{U}$ be a basis of bornology $\mathfrak{B}_{\mathbf{X}}$. We say that a sequence of elements $\mathbf{x}_n \in \mathbf{X}$, $n \in \mathbb{N}$, *$\mathcal{U}$-converges*

[2] in literature we can find also as terms as the *ground state* or *marked element* or *mother wavelet* depending on the context

(or, equivalently, converges bornologically with respect to the bornology $\mathfrak{B}_{\mathbf{X}}$) to $\mathbf{x} \in \mathbf{X}$, if there exists $U \in \mathcal{U}$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $(\mathbf{x}_n - \mathbf{x}) \in U$ for every $n \geq n_0$. We write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{x}_n$. To be more precise, we will sometimes call this the U-convergence of elements from $\mathbf{X}$ to show explicitly which $U \in \mathcal{U}$ we have in the mind.

Remark 2.1 A classical bornology consists of all sets which are bounded in the von Neumann sense, i.e. for a locally convex topological vector space $\mathbf{X}$ equipped with a family of seminorms Q, the set B is *bounded* (or belongs to the von Neumann bornology) if and only if for every $q \in Q$ there exists a constant C_q such that $q(\mathbf{x}) \leq C_q$ for every $\mathbf{x} \in B$.

2.2 Operator spaces

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, and $U_1 \vee U_2 = \text{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(U_1, W_1), (U_2, W_2) \in \mathcal{U} \times \mathcal{W}$, we write $(U_1, W_1) \ll (U_2, W_2)$ if and only if $U_1 \subset U_2$, and $W_1 \supset W_2$.

We use Φ to denote the *class of all functions* $\mathcal{U} \to \mathcal{W}$ *with order* $<$ defined as follows: for $\varphi, \psi \in \Phi$ we write $\varphi < \psi$ whenever $\varphi(U) \subset \psi(U)$ for every $U \in \mathcal{U}$.

Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. We suppose $L(\mathbf{X}, \mathbf{Y}) \subset \Phi$. The bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$ are supposed to be stronger than the corresponding von Neumann bornologies, i.e. the vector operations on the space $L(\mathbf{X}, \mathbf{Y})$ are compatible with the topologies, and the bornological convergence implies the topological convergence. Note that in the terminology [21] the space $L(\mathbf{X}, \mathbf{Y})$ (as an inductive limit of seminormed spaces) is a bornological convex vector space. For a more detail explanation of the structure of $L(\mathbf{X}, \mathbf{Y})$, when both $\mathbf{X}$, $\mathbf{Y}$ are C. B. L. C. S., cf. [15]. For the topological and bornological methods of functional analysis in connection with operators, cf. [26].

2.3 Basic set structures

Let $T \neq \emptyset$ be a set. Denote by 2^T the potential set of the set T and by $\Delta \subset 2^T$ a δ-ring of sets. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ denotes the σ-algebra generated by the system $\mathcal{A}$. Denote by $\Sigma = \sigma(\Delta)$. We use χ_E to denote the characteristic function of the set E. By $p_U : \mathbf{X} \to [0, \infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$ (if U does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_U(\mathbf{x}) = \infty$.). Similarly, p_W denotes the Minkowski functional of the set $W \in \mathcal{W}$.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$ denote by $\hat{\mathbf{m}}_{U,W} : \Sigma \to [0, \infty]$ *a* (U, W)*-semivariation* of a charge (= finitely additive measure) $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ given by

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W \left(\sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i \right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in U; i = 1, 2, \dots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \dots, I\}$. It is well-known that $\hat{\mathbf{m}}_{U,W}$ is a submeasure, i.e. a monotone, subadditive set function with $\hat{\mathbf{m}}_{U,W}(\emptyset) = 0$ for every $(U,W) \in \mathcal{U} \times \mathcal{W}$. The family $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}} = \{\hat{\mathbf{m}}_{U,W};\ (U,W) \in \mathcal{U} \times \mathcal{W}\}$ is said to be the $(\mathcal{U},\mathcal{W})$-semivariation of $\mathbf{m}$.

Denote by $\Delta_{U,W} \subset \Delta$ the largest δ-ring of sets $E \in \Delta$, such that $\hat{\mathbf{m}}_{U,W}(E) < \infty$. Observe that if $(U_1,W_1),(U_2,W_2) \in \mathcal{U} \times \mathcal{W}$ such that $(U_2,W_2) \ll (U_1,W_1)$, then $\Delta_{U_1,W_1} \subset \Delta_{U_2,W_2}$ and therefore $\sigma(\Delta_{U_1,W_1}) \subset \sigma(\Delta_{U_2,W_2})$. Denote by $\Delta_{\mathcal{U},\mathcal{W}} = \{\Delta_{U,W} \subset \Delta;\ (U,W) \in \mathcal{U} \times \mathcal{W}\}$.

Remark 2.2 It is technically convenient to extend the definition of (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$ to an arbitrary subset F of T as follows:

$$\hat{\mathbf{m}}^*_{U,W}(F) = \inf_{E \in \Sigma, F \subset E} \hat{\mathbf{m}}_{U,W}(E).$$

For every $(U,W) \in \mathcal{U} \times \mathcal{W}$, denote by $\|\mathbf{m}\|_{U,W} : \Sigma \to [0,\infty]$ *a scalar* (U,W)-*semivariation* of a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ defined as

$$\|\mathbf{m}\|_{U,W}(E) = \sup \left\| \sum_{i=1}^{I} \lambda_i \mathbf{m}(E \cap E_i) \right\|_{U,W}, \quad E \in \Sigma,$$

where $\|L\|_{U,W} = \sup_{\mathbf{x} \in U} p_W(L(\mathbf{x}))$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \leq 1, i = 1,2,\ldots,I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1,2,\ldots,I\}$. The scalar (U,W)-semivariation $\|\mathbf{m}\|_{U,W}$ is also a submeasure. Denote by $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}} = \{\|\mathbf{m}\|_{U,W}; (U,W) \in \mathcal{U} \times \mathcal{W}\}$.

For every $W \in \mathcal{W}$, denote by $|\mu|_W : \Sigma \to [0,\infty]$ *a* W*-semivariation* of a charge $\mu : \Sigma \to \mathbf{Y}$ given by

$$|\mu|_W(E) = \sup p_W\left(\sum_{i=1}^{I} \lambda_i \mu(E \cap E_i)\right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \leq 1, i = 1,2,\ldots,I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1,2,\ldots,I\}$. The W-semivariation $|\mu|_W$ is also a submeasure. Denote by $\mu_{\mathcal{W}} = \{\mu_W;\ W \in \mathcal{W}\}$.

2.4 Basic convergences of functions

In the theory of integration in Banach spaces we suppose the generalizations of the classical notions, such as almost everywhere convergence, almost uniform convergence, and convergence in measure or semivariation of measurable functions and relations among them as commonly well-known, cf. [9]. All this theory may be generalized to C. B. L. C. S. as follows.

Let $\beta_{\mathcal{U},\mathcal{W}}$ be a lattice of submeasures $\beta_{U,W} : \Sigma \to [0,\infty], (U,W) \in \mathcal{U} \times \mathcal{W}$, where

$$\begin{aligned} \beta_{U_2,W_2} \wedge \beta_{U_3,W_3} &= \beta_{U_2 \wedge U_3, W_2 \vee W_3}, \\ \beta_{U_2,W_2} \vee \beta_{U_3,W_3} &= \beta_{U_2 \vee U_3, W_2 \wedge W_3}, \end{aligned}$$

for $(U_2,W_2),(U_3,W_3) \in \mathcal{U} \times \mathcal{W}$, (e.g. $\beta_{\mathcal{U},\mathcal{W}} = \hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$, or $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}}$).

Denote by $\mathcal{O}(\beta_{U,W}) = \{N \in \Sigma;\ \beta_{U,W}(N) = 0,\ (U,W) \in \mathcal{U} \times \mathcal{W}\}$. The set $N \in \Sigma$ is called $\beta_{\mathcal{U},\mathcal{W}}$-*null* if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(N) = 0$. We say that an assertion holds $\beta_{\mathcal{U},\mathcal{W}}$-*almost everywhere*, shortly $\beta_{\mathcal{U},\mathcal{W}}$-*a.e.*, if it holds everywhere except in a $\beta_{\mathcal{U},\mathcal{W}}$-null set. A set $E \in \Sigma$ is said to be of *finite submeasure* $\beta_{\mathcal{U},\mathcal{W}}$ if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(E) < \infty$.

Definition 2.3 Let $E \in \Sigma$ and $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* (R,E)*-converges* $\beta_{U,W}$-a.e. *to a function* $\mathbf{f} : T \to \mathbf{X}$ if $\lim_{n\to\infty} p_R(\mathbf{f}_n(t) - \mathbf{f}(t)) = 0$ for every $t \in E \setminus N$, where $N \in \mathcal{O}(\beta_{U,W})$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* $(\mathcal{U}, E)$*-converges* $\beta_{\mathcal{U},\mathcal{W}}$-a.e. *to a function* $\mathbf{f} : T \to \mathbf{X}$ if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions (R,E)-converges $\beta_{U,W}$-a.e. to $\mathbf{f}$. We write $\mathbf{f} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n$ $\beta_{\mathcal{U},\mathcal{W}}$-a.e.

If $E = T$, then we will simply say that the sequence R-converges $\beta_{U,W}$-a.e., resp. $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-a.e.

Definition 2.4 Let $E \in \Sigma$ and $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence of functions* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, (R,E)*-converges uniformly to a function* $\mathbf{f} : T \to \mathbf{X}$, if $\lim_{n\to\infty} \|\mathbf{f}_n - \mathbf{f}\|_{E,R} = 0$, where $\|\mathbf{f}\|_{E,R} = \sup_{t\in E} p_R(\mathbf{f}(t))$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* (R,E)*-converges* $\beta_{U,W}$*-almost uniformly to a function* $\mathbf{f} : T \to \mathbf{X}$ if for every $\varepsilon > 0$ there exists a set $N \in \Sigma$, such that $\beta_{U,W}(N) < \varepsilon$ and the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions $(R, E \setminus N)$-converges uniformly to $\mathbf{f}$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* $(\mathcal{U}, E)$*-converges* $\beta_{\mathcal{U},\mathcal{W}}$*-almost uniformly to a function* $\mathbf{f} : T \to \mathbf{X}$, if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions (R,E)-converges $\beta_{U,W}$-almost uniformly to $\mathbf{f}$.

If $E = T$, then we will simply say that the sequence of functions R-converges uniformly, resp. R-converges $\beta_{U,W}$-almost uniformly, resp. $\mathcal{U}$-converges $\beta_{\mathcal{U},\mathcal{W}}$-almost uniformly.

Definition 2.5 Let $E \in \Sigma$ and $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* $\hat{\mathbf{m}}_{U,W}$*-*(R,E)*-converges to a function* $\mathbf{f} : T \to \mathbf{X}$, if for every $\varepsilon > 0$, $\delta > 0$ there exists $n(\varepsilon, \delta) \in \mathbb{N}$, such that for every $n \geq n(\varepsilon,\delta)$, $n \in \mathbb{N}$, holds: $\hat{\mathbf{m}}^*_{U,W}(\{t \in E; p_R(\mathbf{f}_n(t) - \mathbf{f}(t)) \geq \delta\}) < \varepsilon$.

We say that *a sequence* $\mathbf{f}_n : T \to \mathbf{X}$, $n \in \mathbb{N}$, *of functions* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-*$(\mathcal{U}, E)$*-converges to a function* $\mathbf{f} : T \to \mathbf{X}$ if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions $\hat{\mathbf{m}}_{U,W}$-(R,E)-converges to $\mathbf{f}$.

If $E = T$, then we will simply say that the sequence of functions $\hat{\mathbf{m}}_{U,W}$-R-converges, resp. $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-$\mathcal{U}$-converges.

For a more detail explanation of described convergences of functions in C. B. L. C. S. and relations among them, cf. [14].

2.5 Measure structures

For $(U,W) \in \mathcal{U} \times \mathcal{W}$ we say that a charge $\mathbf{m}$ is of *σ-finite (U,W)-semivariation* if there exist sets $E_n \in \Delta_{U,W}$, $n \in \mathbb{N}$, such that $T = \bigcup_{n=1}^{\infty} E_n$. For $\varphi \in \Phi$, we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$, the charge $\mathbf{m}$ is of *σ-finite $(U, \varphi(U))$-semivariation*.

Definition 2.6 We say that a charge $\mathbf{m}$ is *of σ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$ such that for every $U \in \mathcal{U}$ the charge is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation.

In what follows we suppose the charge $\mathbf{m}$ is of σ-finite $(\mathcal{U}, \mathcal{W})$-semivariation. To be more exact we will sometimes specify that a charge $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation to indicate that φ is that function in Definition 2.6 which provides the σ-finiteness of the $(\mathcal{U}, \mathcal{W})$-semivariation of $\mathbf{m}$.

Lemma 2.7 [cf. [17], Lemma 2.2] *Let $\varphi, \psi \in \Phi$ such that $\varphi < \psi$. If a charge $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, then $\mathbf{m}$ is also σ_ψ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.*

If $U \in \mathcal{U}$, $\varphi \in \Phi$, and $\sigma_F(\Delta_{U,\varphi(U)})$ is the smallest local σ-ring of all sets of σ-finite $(U, \varphi(U))$-semivariation (i.e. the following implication is true: if $A \in \Delta_{U,\varphi(U)}$, $B \in \sigma_F(\Delta_{U,\varphi(U)})$, then $A \cap B \in \Delta_{U,\varphi(U)}$), and $\mathcal{O}_F(\hat{\mathbf{m}}_{U,\varphi(U)}) = \mathcal{O}(\hat{\mathbf{m}}_{U,\varphi(U)})$, where $\mathcal{O}_F(\hat{\mathbf{m}}_{U,\varphi(U)}) = \{N \in \sigma_F(\Delta_{U,\varphi(U)}); \hat{\mathbf{m}}_{U,\varphi(U)}(N) = 0\}$.

Lemma 2.8 [cf. [17], Lemma 2.3] *Let $\varphi \in \Phi$. If a charge $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, then $\Sigma = \sigma_F(\Delta_{U,\varphi(U)})$ for every $U \in \mathcal{U}$.*

Let $W \in \mathcal{W}$. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a (W, σ)-additive vector measure, if μ is a $\mathbf{Y}_W$-valued (countable additive) vector measure.

Definition 2.9 We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *$(\mathcal{W}, \sigma)$-additive vector measure*, if there exists $W \in \mathcal{W}$ such that μ is a (W, σ)-additive vector measure.

Note that if $W \in \mathcal{W}$ and $\mu : \Sigma \to \mathbf{Y}$ is a (W, σ)-additive vector measure, then μ is a (W_1, σ)-additive vector measure whenever $W \subset W_1$, $W_1 \in \mathcal{W}$.

We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a $(\mathcal{W}, \sigma)$-*continuous measure* if there exists $W_0 \in \mathcal{W}$, such that $E_n \supset E_{n+1}$, $E_n \in \Sigma$, $n \in \mathbb{N}$, $p_{W_0}(\mu(E_1))) < \infty$, $\bigcap_{n=1}^{\infty} E_n = \emptyset$ implies

$$\lim_{n\to\infty} p_{W_0}(\mu(E_n)) = 0. \tag{1}$$

Note that (1) holds for every $W \supset W_0$, $W \in \mathcal{W}$.

Lemma 2.10 *If μ is a $(\mathcal{W}, \sigma)$-additive measure, then μ is a $(\mathcal{W}, \sigma)$-continuous measure.*

Proof. Since μ is a $(\mathcal{W}, \sigma)$-additive measure, then there exists $W_+ \in \mathcal{W}$, such that μ is a (W_+, σ)-additive measure. Also the measure μ is (W, σ)-additive for every $W \supset W_+$, $W \in \mathcal{W}$.

Let $E_n \supset E_{n+1}$, $E_n \in \Sigma$, $n \in \mathbb{N}$, such that $\bigcap_{n=1}^{\infty} E_n = \emptyset$ and $p_{W_*}(\mu(E_1)) < \infty$ for some $W_* \in \mathcal{W}$. Then $p_W(\mu(E_1)) < \infty$ for every $W \supset W_*$, $W \in \mathcal{W}$.

Put $W_0 = W_+ \vee W_*$. Show that (1) holds. Denote by $F_n = E_n \setminus E_{n+1}$, $n \in \mathbb{N}$. By the (W, σ)-additivity of μ we have

$$0 = \lim_{n\to\infty} p_{W_0}\left(\mu(E_1) - \mu\left(\bigcup_{i=1}^{n} F_i\right)\right) = \lim_{n\to\infty} p_{W_0}(\mu(E_{n+1}).$$

□

Let $W \in \mathcal{W}$ and let $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, be a sequence of (W, σ)-additive vector measures. If for every $\varepsilon > 0$, $E \in \Sigma$ with $p_W(\nu_n(E)) < \infty$, and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i, j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$ such that for every $J \geq J_0$,

$$p_W\left(\nu_n\left(\bigcup_{i=J+1}^{\infty} E_i \cap E\right)\right) < \varepsilon$$

uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures ν_n, $n \in \mathbb{N}$, is *uniformly* (W, σ)*-additive on* Σ.

Note that if a sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W, σ)-additive on Σ for $W \in \mathcal{W}$, then the sequence ν_n, $n \in \mathbb{N}$, of measures is uniformly (W_1, σ)-additive on Σ whenever $W_1 \supset W$, $W_1 \in \mathcal{W}$.

Definition 2.11 We say that the family of measures $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, is *uniformly* $(\mathcal{W}, \sigma)$*-additive on* Σ, if there exists $W \in \mathcal{W}$ such that the family of measures ν_n, $n \in \mathbb{N}$, is uniformly (W, σ)-additive on Σ.

The following definition generalizes the notion of the σ-additivity of an operator valued measure in the strong operator topology in Banach spaces, cf. [9], to C. B. L. C. S.

Definition 2.12 Let $\varphi \in \Phi$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_φ*-additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$, the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\varphi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ*-additive (bornological) measure* if there exists $\varphi \in \Phi$, such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows a charge $\mathbf{m}$ is supposed to be σ-additive bornological measure. If $\varphi < \psi$, $\varphi, \psi \in \Phi$, and a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is a σ_φ-additive measure, then $\mathbf{m}$ is a σ_ψ-additive measure. Indeed, the fact that $\mathbf{m}$ is of σ_ψ-finite $(\mathcal{U}, \mathcal{W})$-semivariation follows from Lemma 2.7. The assertion that for every $A \in \Delta_{U,W}$, the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\psi(U), \sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, is implied from the inequality $p_{\psi(U)}(\mathbf{y}) \le p_{\varphi(U)}(\mathbf{y})$, $\mathbf{y} \in \mathbf{Y}$.

2.6 An integral in C. B. L. C. S.

We use $\mathcal{M}_{\Delta,\mathcal{U}}$ to denote *the space of all* $(\Delta, \mathcal{U})$*-measurable functions*, i.e. the largest vector space of functions $\mathbf{f} : T \to \mathbf{X}$ with the property: there exists $R \in \mathcal{U}$, such that for every $U \in \mathcal{U}$, $U \supset R$, and $\delta > 0$ the set $\{t \in T; \, p_U(\mathbf{f}(t)) \ge \delta\} \in \Sigma$. In what follows we deal only with $(\Delta, \mathcal{U})$-measurable functions. Note that in this $\hat{\mathbf{m}}^*_{U,W} = \hat{\mathbf{m}}_{U,W}$ for every $(U, W) \in \mathcal{U} \times \mathcal{W}$ in Definition 2.5.

Definition 2.13 A function $\mathbf{f} : T \to \mathbf{X}$ is called Δ*-simple* if $\mathbf{f}(T)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x}) \in \Delta$ for every $\mathbf{x} \in \mathbf{X} \setminus \{0\}$. Let $\mathcal{S}$ denote the space of all Δ-simple functions.

For $(U, W) \in \mathcal{U} \times \mathcal{W}$, a function $\mathbf{f} : T \to \mathbf{X}$ is said to be $\Delta_{U,W}$*-simple* if $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, where $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, such that $E_i \cap E_j = \emptyset$, for $i \ne j$, $i, j = 1, 2, \ldots, I$. The space of all $\Delta_{U,W}$-simple functions is denoted by $\mathcal{S}_{U,W}$.

A function $\mathbf{f} \in \mathcal{S}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$*-simple* if there exists a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that $\mathbf{f} \in \mathcal{S}_{U,W}$. The space of all $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions is denoted by $\mathcal{S}_{\mathcal{U},\mathcal{W}}$.

For every $E \in \Sigma$ and $\mathbf{f} \in \mathcal{S}_{U,W}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, we define the integral by the formula $\int_E \mathbf{f} \, d\mathbf{m} = \sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i$, where $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, $E_i \cap E_j = \emptyset$, $i \ne j$, $i, j = 1, 2, \ldots, I$. Note that for the function $\mathbf{f} \in \mathcal{S}_{U,W}$ the integral $\int_\cdot \mathbf{f} \, d\mathbf{m}$ is a (W, σ)-additive measure on Σ.

The following result is a version of the classical Vitali-Hahn-Saks-Nikodym theorem in our setting, cf. [8].

Theorem 2.14 *Let $\gamma_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, be $(\mathcal{W},\sigma)$-additive measures and let $\mathcal{W}\text{-}\lim_{n\to\infty} \gamma_n(E) = \gamma(E)$ exists in $\mathbf{Y}$ for each $E \in \Sigma$. Then γ_n, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ, and consequently, γ is a $(\mathcal{W},\sigma)$-additive measure on Σ.*

Theorem 2.15 [cf. [17], Theorem 3.8] *Let $\mathbf{m}$ be a σ-additive measure and $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$. If there exists a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions, such that*

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e.,

(b) the integrals $\int_{\cdot} \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ,

then the limit $\nu(E,\mathbf{f}) = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}$ exists uniformly in $E \in \Sigma$.

Definition 2.16 A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is said to be *$\Delta_{\mathcal{U},\mathcal{W}}$-integrable* if there exists a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions, such that

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e.,

(b) $\int_{\cdot} \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W},\sigma)$-additive measures on Σ.

Let $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ denote *the family of all $\Delta_{\mathcal{U},\mathcal{W}}$-integrable functions*. Then *the integral* of a function $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ on a set $E \in \Sigma$ is defined by the equality

$$\mathbf{y}_E = \int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \mathcal{W}\text{-} \lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}.$$

Theorem 2.17 [cf. [17], Theorem 4.2] *Let $\nu(E,\mathbf{f}) = \int_E \mathbf{f} \, \mathrm{d}\mathbf{m}$, $E \in \Sigma$ and $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$. Then $\nu(\cdot,\mathbf{f}) : \Sigma \to \mathbf{Y}$ is a $(\mathcal{W},\sigma)$-additive measure.*

The following theorem gives a criterium of integrability of a $(\Delta,\mathcal{U})$-measurable function.

Theorem 2.18 [cf. [17], Theorem 4.3] *A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is $\Delta_{\mathcal{U},\mathcal{W}}$-integrable if and only if there exists a sequence $\mathbf{f}_n \in \mathcal{S}_{\mathcal{U},\mathcal{W}}$, $n \in \mathbb{N}$, of functions such that*

(a) $(\mathcal{U},E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to $\mathbf{f}$, and

(b) the limit $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m} = \nu(E)$ exists for every $E \in \Sigma$.

In this case $\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}$ for every set $E \in \Sigma$ and this limit is uniform on Σ.

For $(U,W) \in \mathcal{U} \times \mathcal{W}$ denote by $\mathcal{I}_{U,W}$ the set of all Dobrakov's integrable functions with respect to Banach spaces $\mathbf{X}_U$, $\mathbf{Y}_W$. Observe that the space of all integrable functions is constructed as a union of the net

$$\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta} = \bigcup_{(U,W)\in\mathcal{U}\times\mathcal{W}} \mathcal{I}_{U,W}.$$

Observe that the family $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ (a collection of all $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null sets) connects the whole theory of the integration by the notions of the type "$\mathcal{U},\mathcal{W}$-almost everywhere" through the rows of the integrable functions $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ into the one unit. In [16], Lemma 3.13, it is proved that $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ is an ideal of subsets of Σ. So, the ideal $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ gives a new quality to the theory of integration in the bornological spaces, which we do not observe in the classical case or in the integration in Banach spaces where the structure of null set has no sense. Moreover, in every previous theorem we can consider generally the set from the $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ in the assertions of the type "$\hat{\mathbf{m}}_{U,W}$-almost everywhere" for $(U,W) \in \mathcal{U} \times \mathcal{W}$.

3 Theorems on integrable functions

Lemma 3.1 *Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. Then*

$$\hat{\mathbf{m}}_{U,W}(E) = \sup \left\{ p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right) ; \, \mathbf{f} \in \mathcal{S}_{U,W}, \|\mathbf{f}\|_{E,U} \leq 1 \right\}$$

for every set $E \in \sigma(\Delta_{U,W})$. Hence for every $\mathbf{f} \in \mathcal{S}_{U,W}$ and every set $E \in \sigma(\Delta_{U,W})$ the inequality

$$p_W \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right) \leq \|\mathbf{f}\|_{E,U} \cdot \hat{\mathbf{m}}_{U,W}(E)$$

holds.

The proof is trivial and therefore omitted. Note that the assertion of Lemma 3.1 holds also when replacing $\Delta_{U,W}$-simple functions by $\Delta_{U,W}$-integrable functions as it is proved in [18]. The following result is a useful one.

Theorem 3.2 *If $\mathbf{f}$ is a $\Delta_{\mathcal{U},\mathcal{W}}$-integrable function and φ is a bounded scalar $(\Delta, \mathcal{U})$-measurable function, then $\varphi\,\mathbf{f}$ is a $\Delta_{\mathcal{U},\mathcal{W}}$-integrable function.*

Proof. By Definition 2.16 there exists $U \in \mathcal{U}$, $(R,S) \in \mathcal{U} \times \mathcal{W}$, and $M \in \Sigma$ such that $\hat{\mathbf{m}}_{R,S}(M) = 0$ and $\lim_{n\to\infty} p_U(\mathbf{f}_n(t) - \mathbf{f}(t)) = 0$ for every $t \in T \setminus M$. Without loss of generality suppose that the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of functions U-converges to $\mathbf{f}$. Since $\mathbf{m}$ is a σ_{φ_1}-additive measure for some $\varphi_1 \in \Phi$, for U there exists $W_1 \in \mathcal{W}$ such that $\varphi_1(U) = W_1$. By definition, there exists $W_2 \in \mathcal{W}$ such that the integrals $\int_\cdot \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly (W_2, σ)-additive measures on Σ. Put $\varphi(U) = W = W_1 \vee W_2$. Then the integrals $\int_\cdot \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly (W, σ)-additive measures on Σ and the measure $\mathbf{m}$ is also of φ-finite $(\mathcal{U}, \mathcal{W})$-semivariation.

Without loss of generality suppose $|\varphi(t)| \leq 1$ for every $t \in T$ and choose the scalar $\Delta_{U,W}$-simple functions φ_n, $n \in \mathbb{N}$, converging on the whole T to φ, for which $\|\varphi_n\|_{T,U} \leq 1$ for every $n \in \mathbb{N}$. Then $\varphi_n \, \mathbf{f}_n$, $n \in \mathbb{N}$, is a sequence of $\Delta_{U,W}$-simple functions U-converging $\hat{\mathbf{m}}_{U,W}$-a.e. to the function $\varphi\,\mathbf{f}$. From the definition of the scalar (U,W)-semivariation we get the inequality

$$\left\| \int_\cdot \varphi_n \mathbf{f}_n \, \mathrm{d}\mathbf{m} \right\|_{U,W} (E) \leq \left\| \int_\cdot \mathbf{f}_n \, \mathrm{d}\mathbf{m} \right\|_{U,W} (E)$$

for every set $E \in \Sigma$ and every $n \in \mathbb{N}$.

Since the integrals $\int_\cdot \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly (W, σ)-additive measures on Σ, from the Hahn-Banach theorem and definition of the scalar (U,W)-semivariation we immediately obtain the uniform continuity of $\| \int_\cdot \mathbf{f}_n \, \mathrm{d}\mathbf{m}\|_{U,W}(\cdot)$.

So, the integrals $\int_\cdot \varphi_n \mathbf{f}_n \,d\mathbf{m}$, $n \in \mathbb{N}$, are uniformly σ-additive $\mathbf{Y}_W$-valued measures on Σ. This completes the proof of the integrability of the function $\varphi\,\mathbf{f}$. $\square$

Remark 3.3 For the sake of brevity we will continue proving each ongoing theorem only using a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$ because the case (U_1, W_1), $(U_2, W_2) \in \mathcal{U} \times \mathcal{W}$ such that $U_1 \neq U_2$ or $W_1 \neq W_2$ may be reduced to this case putting $U = U_1 \vee U_2$ and $W = \varphi(U)$, where $\varphi(U) = \varphi_1(U) \vee W_1 \vee W_2$, $\varphi \in \Phi$, and $\varphi_1 \in \Phi$ is such that T is of σ_{φ_1}-finite $(U, \varphi_1(U))$-semivariation.

Theorem 3.4 *Let $A \in \Delta_{\mathcal{U},\mathcal{W}}$ and $\mathbf{f} : T \to \mathbf{X}$ be a $\mathcal{U}$-bounded $(\Delta, \mathcal{U})$-measurable function. If there exists a sequence of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions $\mathbf{f}_n$, $n = 1, 2, \ldots$, $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-$\mathcal{U}$-converging to the function $\mathbf{f}\chi_A$, then $\mathbf{f}\chi_A \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$.*

Proof. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. For each $k = 1, 2, \ldots$, take an n_k such that $\hat{\mathbf{m}}_{U,W}(A_k) < 2^{-k}$ where

$$A_k = \left\{ t \in A;\; p_U(\mathbf{f}_{n_k}(t) - \mathbf{f}(t)) > \frac{1}{2^k} \right\}.$$

Put $B_k = \bigcup_{i=k}^{\infty} A_i$ for $k = 1, 2, \ldots$, and let $B = \bigcap_{k=1}^{\infty} B_k$. Then by σ-subadditivity of the (U, W)-semivariation $\hat{\mathbf{m}}_{U,W}$ we have that $B \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$ and therefore $\int_E \mathbf{f}\chi_B \,d\mathbf{m} = 0$ for each $E \in \Sigma$.

For each $k = 1, 2, \ldots$, put $\mathbf{f}'_k = \mathbf{f}_{n_k}\chi_{A\setminus B_k}$. Then $\mathbf{f}'_k$, $k = 1, 2, \ldots$, is a sequence of $\Delta_{U,W}$-simple functions U-converging on the whole T to the function $\mathbf{f}\chi_{A\setminus B}$ and at the same time $\|\mathbf{f}'_k\|_{T,U} \le \|\mathbf{f}\|_{T,U} + 1$ for each $k = 1, 2, \ldots$. Further, for each set $E \in \Sigma$ and each $k_1, k_2 \in \mathbb{N}$ such that $k_1 \le k_2$ the following inequalities hold

$$\begin{aligned} p_W\left(\int_E \mathbf{f}'_{k_1}\,d\mathbf{m} - \int_E \mathbf{f}'_{k_2}\,d\mathbf{m}\right) &\le p_W\left(\int_{E\cap(A\setminus B_{k_1})} (\mathbf{f}'_{k_1} - \mathbf{f}'_{k_2})\,d\mathbf{m}\right) \\ &\quad + p_W\left(\int_{E\cap B_{k_1}} \mathbf{f}'_{k_1}\,d\mathbf{m}\right) + p_W\left(\int_{E\cap B_{k_1}} \mathbf{f}'_{k_2}\,d\mathbf{m}\right) \\ &\le \frac{1}{2^{k_1-2}} \cdot (\hat{\mathbf{m}}_{U,W}(A) + \|\mathbf{f}\|_{T,U} + 1). \end{aligned}$$

By the assumption of the theorem we have $\hat{\mathbf{m}}_{U,W}(A) + \|\mathbf{f}\|_{T,U} < \infty$ and since $\mathbf{Y}_W$ is complete, then from the above inequalities the existence of the limit $W\text{-}\lim_{k\to\infty} \int_E \mathbf{f}'_k\,d\mathbf{m} = \nu(E) \in \mathbf{Y}_W$ follows for each $E \in \Sigma$. From here the integrability of $\mathbf{f}\chi_{A\setminus B}$ and consequently of $\mathbf{f}\chi_A$ follows by Theorem 2.18. $\square$

Definition 3.5 We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is of *continuous $(\mathcal{U}, \mathcal{W})$-semivariation* if for every couple $(U, W) \in \mathcal{U} \times \mathcal{W}$,

$$E_n \supset E_{n+1}, E_n \in \Sigma, n \in \mathbb{N}, \hat{\mathbf{m}}_{U,W}(E_1) < \infty, \bigcap_{n=1}^{\infty} E_n = \emptyset \;\Rightarrow\; \hat{\mathbf{m}}_{U,W}(E_n) \searrow 0.$$

Theorem 3.6 *Let a charge $\mathbf{m}$ be of continuous $(\mathcal{U}, \mathcal{W})$-semivariation. If $A \in \Delta_{\mathcal{U},\mathcal{W}}$ and $\mathbf{f} : T \to \mathbf{X}$ is a $\mathcal{U}$-bounded $(\Delta, \mathcal{U})$-measurable function, then $\mathbf{f}\chi_A \in \mathcal{I}_{\mathcal{U},\mathcal{W}}$.*

Proof. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$ and let us consider a sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of $\Delta_{U,W}$-simple functions, such that the sequence $\mathbf{f}_n \chi_A$, $n \in \mathbb{N}$, U-converges on the whole space T to the function $\mathbf{f}\chi_A$ and $\|\mathbf{f}_n\|_{A,U} \leq \|\mathbf{f}\|_{A,U}$ for every $n \in \mathbb{N}$. Since $A \in \Delta_{U,W}$, then the inequality

$$p_W\left(\int_E \mathbf{f}_n \chi_A \, \mathrm{d}\mathbf{m}\right) \leq \|\mathbf{f}\|_{A,U} \cdot \hat{\mathbf{m}}_{U,W}(A \cap E), \quad n \in \mathbb{N},\ E \in \Sigma,$$

and the fact that $\mathbf{m}$ is of continuous (U, W)-semivariation imply the uniform (W, σ)-additivity of integrals $\int_{\cdot} \mathbf{f}_n \chi_A \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, on Σ. This proves the theorem. $\square$

Note that without the assumption $\mathbf{m}$ is of continuous $(\mathcal{U}, \mathcal{W})$-semivariation the previous theorem does not hold even in the case when a charge $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ is σ-additive in the equibornology of the space $L(\mathbf{X}, \mathbf{Y})$ (i.e., if for every $(U, W) \in \mathcal{U} \times \mathcal{W}$ the restriction $\mathbf{m}_{U,W}(E)\mathbf{x} = \mathbf{m}(E)\mathbf{x}$, $E \in \Delta_{U,W}$, of a charge $\mathbf{m}$ to the set system $\Delta_{U,W}$ is a σ-additive vector measure in the uniform topology of the space $L(\mathbf{X}_U, \mathbf{Y}_W)$). Immediately we have the following

Theorem 3.7 *Suppose that* $\mathbf{m}$ *is not of continuous* $(\mathcal{U}, \mathcal{W})$*-semivariation. Then there exists a set* $A \in \Delta_{\mathcal{U},\mathcal{W}}$ *and a* $\mathcal{U}$*-bounded function* $\mathbf{f} : T \to \mathbf{X}$ *such that the function* $\mathbf{f}\chi_A$ *is not* $\Delta_{\mathcal{U},\mathcal{W}}$*-integrable.*

Proof. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. By assumption of theorem there exists an $\varepsilon > 0$, a set $A \in \Delta_{U,W}$ and a sequence of pairwise disjoint sets $E_n \in \Delta_{U,W}$, $n = 1, 2, \ldots$, such that $\hat{\mathbf{m}}_{U,W}(A \cap E_n) > \varepsilon$ for each $n = 1, 2, \ldots$. According to Lemma 3.1 for each $n = 1, 2, \ldots$, there is a $\Delta_{U,W}$-simple function $\mathbf{f}_n$ with

$$\sup_{t \in A \cap E_n} p_U(\mathbf{f}_n(t)) \leq 1$$

such that

$$p_W\left(\int_{A \cap E_n} \mathbf{f}_n \, \mathrm{d}\mathbf{m}\right) > \varepsilon.$$

Put $\mathbf{f} = \sum_{n=1}^{\infty} \mathbf{f}_n \chi_{E_n}$. Clearly, $\mathbf{f}$ is a U-bounded function. According to Theorem 2.17 the indefinite integral $E \to \int_E \mathbf{f} \, \mathrm{d}\mathbf{m}$ of a $\Delta_{U,W}$-integrable function $\mathbf{f}$ for $E \in \Sigma$ is a (W, σ)-additive vector measure on Σ, and therefore the function $\mathbf{f}\chi_A$ cannot be $\Delta_{U,W}$-integrable. $\square$

4 Convergence theorems

In this part of paper we prove some convergence theorems for our integral (including fundamental theorem on interchange of limit and the integral, see Theorem 4.4) and we give a characterization of the set of all integrable functions $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$, see Corollary 4.5. The following lemma will be useful, cf. [9].

Lemma 4.1 *If a sequence* $\mathbf{f}_n \in \mathcal{M}_{\Delta,\mathcal{U}}$*,* $n = 1, 2, \ldots,$ *of functions* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-*$\mathcal{U}$*-converges to a function* $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ *on every set* $E \in \Delta$*, then there exists a subsequence* $\mathbf{f}_{n_k}$*,* $k = 1, 2, \ldots,$ *of* $\mathbf{f}_n$ $\mathcal{U}$*-converging* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e. on the whole* T *to the function* $\mathbf{f}$*.*

Proof. Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. Let $\mathbf{f}_0 = \mathbf{f}$ and take a set $B \in \Sigma$, $B \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$, such that $\mathbf{f}_n \chi_{T \setminus B}$, $n = 0, 1, 2, \ldots$, are $\Delta_{U,W}$-measurable. Let $N(\mathbf{f}) = \{t \in T;\, \mathbf{f}(t) \neq 0\}$ and put

$$F = \bigcup_{n=0}^{\infty} (T \setminus B) \cap N(\mathbf{f}_n).$$

Clearly, $F \in \sigma(\Delta_{U,W})$. Choose an increasing sequence $F_k \subset \Delta_{U,W}$, $k = 1, 2, \ldots$, such that $F = \bigcup_{k=1}^{\infty} F_k$. By assumption and by [9], § 1.2, there exist a subsequence $\mathbf{f}_{1,i}$, $i = 1, 2, \ldots$, of $\mathbf{f}_n$, a set $A_1 \in \sigma(\Delta_{U,W}) \cap F_1$, $A_1 \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$ and a $\Delta_{U,W} \cap (F_1 \setminus A_1)$-measurable function $\mathbf{g}$ such that $\mathbf{f}_{1,i}$ U-converges $\hat{\mathbf{m}}_{U,W}$-almost uniformly to $\mathbf{g}$ in F_1. Adapting proofs of Theorem 22.B and 22.C in [12] we conclude that $\mathbf{f} = \mathbf{g}$ $\hat{\mathbf{m}}_{U,W}$-a.e. in F_1 and consequently, there exists $A_1' \subset F_1$, $A_1' \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$ with $A_1 \in \sigma(\Delta_{U,W})$, such that $\mathbf{f}_{1,i}$ U-converges on $F_1 \setminus A_1'$ to $\mathbf{f}$.

Repeating the argument with the subsequence $\mathbf{f}_{1,i}$, $i = 1, 2, \ldots$, we get a subsequence $\mathbf{f}_{2,i}$, $i = 1, 2, \ldots$, of $\mathbf{f}_{1,i}$, and a set $A_2' \subset F_2$, $A_2' \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$ with $A_2 \in \sigma(\Delta_{U,W})$, such that $\mathbf{f}_{2,i}$ U-converges on $F_2 \setminus N_2'$ to $\mathbf{f}$. Repeating this process n-times we obtain a subsequence $\mathbf{f}_{n,i}$, $i = 1, 2, \ldots$, of $\mathbf{f}_{n-1,i}$, and a set $A_n' \subset F_n$, $A_n' \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$ with $A_n \in \sigma(\Delta_{U,W})$, such that $\mathbf{f}_{n,i}$ U-converges on $F_n \setminus A_n'$ to $\mathbf{f}$. Put $A = \bigcup_{n=1}^{\infty} A_n'$. Then clearly $A \in \sigma(\Delta_{U,W})$, $A \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$ and the sequence $\mathbf{f}_{n,n}$, $n = 1, 2, \ldots$, U-converges on $F \setminus A$ to $\mathbf{f}$. Since $T = F \cup B$ and $(B \cup A) \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$, the assertion of lemma follows. □

Theorem 4.2 *Let $\mathbf{f}$ be a $(\Delta, \mathcal{U})$-measurable function and let the sequence $\mathbf{f}_n$, $n \in \mathbb{N}$, of $\Delta_{\mathcal{U},\mathcal{W}}$-integrable functions $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-$\mathcal{U}$-converge to $\mathbf{f}$ on every set $E \in \Delta$. Then the following conditions are equivalent:*

(i) for every set $E \in \Sigma$ there exists a limit

$$\mathcal{W}\text{-}\lim_{n \to \infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m} = \nu(E) \in \mathbf{Y};$$

(ii) the integrals $\int_{\cdot} \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W}, \sigma)$-additive measures on Σ;

(iii) the limit $\mathcal{W}\text{-}\lim_{n \to \infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m} = \nu(E)$ exists in $\mathbf{Y}$ uniformly with respect to $E \in \Sigma$.

If anyone of these conditions holds, then the function $\mathbf{f}$ is $\Delta_{\mathcal{U},\mathcal{W}}$-integrable with $\int_E \mathbf{f} \mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n \to \infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}$ for every set $E \in \Sigma$ and this limit is uniform on $E \in \Sigma$. Moreover, $\nu : \Sigma \to \mathbf{Y}$ is a $(\mathcal{W}, \sigma)$-additive measure.

Proof. The fact that the set functions $\int_{\cdot} \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are $(\mathcal{W}, \sigma)$-additive on Σ is obvious for $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions $\mathbf{f}_n$, $n \in \mathbb{N}$. By Theorem 2.17 the same is true for $\Delta_{\mathcal{U},\mathcal{W}}$-integrable functions $\mathbf{f}_n$, $n \in \mathbb{N}$. Then by Theorem 2.14 (i)⇒(ii) and obviously (iii)⇒(i).

Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. Let (ii) hold and suppose the contrary, i.e. (ii)$\not\Rightarrow$(iii). Then there would exist an $\varepsilon > 0$, a subsequence of natural numbers n_k and a sequence $E_k \subset \Sigma$, $k = 1, 2, \ldots$, such that

$$p_W \left(\int_{E_k} \mathbf{f}_{n_k} \, \mathrm{d}\mathbf{m} - \int_{E_k} \mathbf{f} \, \mathrm{d}\mathbf{m} \right) \geq \varepsilon$$

for $k = 1, 2, \ldots$. On the other hand, by Lemma 4.1 there exists a subsequence $\mathbf{f}_{n_{k_r}}$, $r = 1, 2, \ldots$, of $\mathbf{f}_{n_k}$ such that $\mathbf{f}_{n_{k_r}}$ U-converges $\hat{\mathbf{m}}_{U,W}$-a.e. to $\mathbf{f}$ in T. Then there exists r_0 such that

$$p_W\left(\int_E \mathbf{f}_{n_{k_r}}\, d\mathbf{m} - \int_E \mathbf{f}\, d\mathbf{m}\right) < \varepsilon$$

for all $r \geq r_0$ and every $E \in \Sigma$. This contradiction shows the implication (ii)$\Rightarrow$(iii). Therefore these conditions are equivalent.

By Lemma 4.1 there exists a subsequence $\mathbf{f}_{n_k}$, $k = 1, 2, \ldots$, of $\mathbf{f}_n$ such that $\mathbf{f}_{n_k}$ U-converges $\hat{\mathbf{m}}_{U,W}$-a.e. to $\mathbf{f}$ in T. Then $\mathbf{f}$ is $\Delta_{U,W}$-integrable and

$$\int_E \mathbf{f}\, d\mathbf{m} = W\text{-}\lim_{k\to\infty} \int_E \mathbf{f}_{n_k}\, d\mathbf{m} = W\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\, d\mathbf{m}$$

for $E \in \Sigma$. By (iii) the limit is uniform on Σ. The last statement follows from Theorem 2.14. □

The following theorem answers the question on enlargement procedure to the space $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$.

Theorem 4.3 *If a sequence $\mathbf{f}_n \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ of functions $\mathcal{U}$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to a function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ and the integrals $\int_\cdot \mathbf{f}_n\, d\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W}, \sigma)$-additive measures on Σ, then $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ and*

$$\int_E \mathbf{f}\, d\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\, d\mathbf{m}$$

for every set $E \in \Sigma$ and this limit is uniform on Σ.

Proof. By Theorem 2.15 for each set $E \in \Sigma$ there exists the limit

$$\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\, d\mathbf{m} = \nu(E)$$

and this limit is uniform on Σ. Therefore it is enough to prove that for $(U, W) \in \mathcal{U} \times \mathcal{W}$ the function $\mathbf{f}$ is $\Delta_{U,W}$-integrable and that $\nu(E) = \int_E \mathbf{f}\, d\mathbf{m}$ for every set $E \in \Sigma$.

Since $\mathbf{f}$ is a $\Delta_{U,W}$-measurable function, take a sequence $\mathbf{h}_n$, $n \in \mathbb{N}$, of $\Delta_{U,W}$-simple functions U-converging to $\mathbf{f}$ on the whole T. Consider the sequence $\mathcal{K} = \{\mathbf{f}_1, \mathbf{h}_1, \mathbf{f}_2, \mathbf{h}_2, \ldots, \mathbf{f}_n, \mathbf{h}_n, \ldots\}$ and put

$$F = \bigcup_{n=1}^{\infty} \Big\{t \in T;\, p_U(\mathbf{f}_n(t)) + p_U(\mathbf{h}_n(t)) > 0\Big\}.$$

Using notation from [9], Theorem 1, there exists a set $N \in \sigma(\Delta_{U,W})$, $N \subset F$, and a nondecreasing sequence of the sets $F_{j,k} \in \Delta_{U,W}$, $j, k \in \mathbb{N}$, with $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} F_{j,k} = F \setminus N$, such that

$$\int_E \mathbf{f}_n \chi_N\, d\mathbf{m} = \int_E \mathbf{h}_n \chi_N\, d\mathbf{m} = 0$$

for every $E \in \Sigma$ and $n \in \mathbb{N}$, and, moreover, the sequence $\mathcal{K}$ converges U-uniformly to the function $\mathbf{f}$ on every set $F_{j,k}$, $j,k \in \mathbb{N}$. Choose a subsequence $n_{j,k}$, $j,k \in \mathbb{N}$, such that for every couple (j,k) there is

$$\|\mathbf{h}_{n_{j,k}} - \mathbf{f}_{n_{j,k}}\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) < \frac{1}{2^j k}.$$

Put $\mathbf{g}_{j,k} = \mathbf{h}_{n_{j,k}}\chi_N + \mathbf{h}_{n_{j,k}}\chi_{F_{j,k}}$ for every (j,k). Then $\mathbf{g}_{j,k}$, $j,k \in \mathbb{N}$, is a sequence of $\Delta_{U,W}$-simple functions U-converging $\hat{\mathbf{m}}_{U,W}$-a.e. on the whole T to the function $\mathbf{f}$ for every $j \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ and every set $E \in \Sigma$ there holds

$$\begin{aligned} p_W\left(\nu(E) - \int_E \sum_{j=1}^{J} \mathbf{g}_{j,k}\,\mathrm{d}\mathbf{m}\right) &\le p_W\left(\sum_{j=1}^{J} \int_{E\cap F_{j,k}} (\mathbf{g}_{j,k} - \mathbf{f}_{n_{j,k}})\,\mathrm{d}\mathbf{m}\right) \\ &\quad + p_W\left(\sum_{j=1}^{J} \int_{E\cap(F\setminus N\setminus F_{j,k})} \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}\right) \\ &\quad + p_W\left(\nu(E) - \int_E \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}\right). \end{aligned}$$

Consequently by Lemma 3.1 and definition of W-semivariation we have

$$\begin{aligned} p_W\left(\nu(E) - \int_E \sum_{j=1}^{J} \mathbf{g}_{j,k}\,\mathrm{d}\mathbf{m}\right) &\le \sum_{j=1}^{J} \|\mathbf{h}_{n_{j,k}} - \mathbf{f}_{n_{j,k}}\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) \\ &\quad + \sum_{j=1}^{J} \left|\int_{\cdot} \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}\right|_W (F\setminus N\setminus F_{j,k}) \\ &\quad + p_W\left(\nu(E) - \int_E \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}\right). \end{aligned}$$

Let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that $\frac{1}{2^j k_0} < \frac{\varepsilon}{3}$, $j = 1,2,\ldots,J$. Since $\nu(E) = W\text{-}\lim_{k\to\infty} \int_E \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}$ uniformly with respect to $E \in \Sigma$ for every $j \in \mathbb{N}$, we may choose $k_1 \ge k_0$ such that

$$p_W\left(\nu(E) - \int_E \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}\right) < \frac{\varepsilon}{3}, \qquad j = 1,2,\ldots,J,$$

for all $k \ge k_1$ and for all $E \in \Sigma$. Thus choosing $k \ge k_1$ we have

$$\|\mathbf{h}_{n_{j,k}} - \mathbf{f}_{n_{j,k}}\|_{F_{j,k},U} \cdot \hat{\mathbf{m}}_{U,W}(F_{j,k}) < \frac{\varepsilon}{3}, \qquad j = 1,2,\ldots,J, \tag{2}$$

and

$$p_W\left(\nu(E) - \int_E \mathbf{f}_{n_{j,k}}\,\mathrm{d}\mathbf{m}\right) < \frac{\varepsilon}{3}, \qquad j = 1,2,\ldots,J, \tag{3}$$

for all $E \in \Sigma$. Since $\int_{\cdot} \mathbf{f}_n\,\mathrm{d}\mathbf{m}$, $n = 1,2,\ldots$, are uniformly (W,σ)-additive measures on Σ, then its W-semivariations $|\int_{\cdot} \mathbf{f}_n\,\mathrm{d}\mathbf{m}|_W$ are uniformly continuous on Σ and since also $(F\setminus N\setminus F_{j,k}) \searrow \emptyset$, then there exists $k_2 \ge k_1$ such that

$\left|\int_{\cdot} \mathbf{f}_n \, \mathrm{d}\mathbf{m}\right|_W (F \setminus N \setminus F_{j,k}) < \frac{\varepsilon}{3}$, $j = 1, 2, \ldots, J$, for all $k \geq k_2$ and for all $n \in \mathbb{N}$. Thus, in particular,

$$\left|\int_{\cdot} \mathbf{f}_{n_{j,k}} \, \mathrm{d}\mathbf{m}\right|_W (F \setminus N \setminus F_{j,k}) < \frac{\varepsilon}{3}, \qquad j = 1, 2, \ldots, J, \tag{4}$$

for $k \geq k_2$. Consequently, from (2), (3) and (4) we obtain

$$p_W \left(\nu(E) - \int_E \sum_{j=1}^{J} \mathbf{g}_{j,k} \, \mathrm{d}\mathbf{m} \right) < \varepsilon$$

for $k \geq k_2$ and $E \in \Sigma$. From it follows that $W\text{-}\lim_{k\to\infty} \int_E \mathbf{g}_k \, \mathrm{d}\mathbf{m} = \nu(E)$ for every $E \in \Sigma$. So by Theorem 2.18 the function $\mathbf{f}$ is $\Delta_{U,W}$-integrable and $\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \nu(E)$ for every $E \in \Sigma$. The proof is complete. $\square$

From the definition of $\Delta_{\mathcal{U},\mathcal{W}}$-integrable functions and from the previous theorem we immediately have that the set $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ is the smallest set of functions containing $\mathcal{S}_{\mathcal{U},\mathcal{W}}$ for which Theorem 4.3 is valid.

Theorem 4.4 (on interchange of limit and integral) *If a sequence* $\mathbf{f}_n \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$, $n \in \mathbb{N}$, *of functions* $\mathcal{U}$*-converges* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e. to a function* $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$, *and for every set* $E \in \Sigma$ *there exists a limit* $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, *then*

- *(a)* $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$;
- *(b)* $\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}$ *for every set* $E \in \Sigma$; *and*
- *(c) the limit* $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m} = \int_E \mathbf{f} \, \mathrm{d}\mathbf{m}$ *exists in* $\mathbf{Y}$ *uniformly with respect to* $E \in \Sigma$.

Proof. The assertion follows directly from Theorem 4.3 and Theorem 2.14. $\square$

Corollary 4.5 From Theorem 2.18 and Theorem 4.4 we immediately obtain the following characterization of the sets of all integrable functions $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$: it is a smallest class of functions which contains the set $\mathcal{S}_{\mathcal{U},\mathcal{W}}$ and for which Theorem 4.4 holds.

Theorem 4.4 shows that if the process of Theorem 2.18 is repeated with sequences of functions in $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ instead of $\mathcal{S}_{\mathcal{U},\mathcal{W}}$ we obtain only $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ and no new $(\Delta,\mathcal{U})$-measurable functions are obtained. In other words, we cannot obtain a larger extension of the integral when repeating the enlargement procedure to the space $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$.

If we reduce situations in [7], [22], [24], and [25] to C. B. L. C. S., we do not obtain greater classes of integrable functions than our integral yields. However, it is only our integral that preserves the theorem of interchange of limit and integral for the class of all integrable functions and hence it can be considered as the complete generalization of the Dobrakov's integral theory which cannot be said about each above noted integrals.

As a consequence of Theorem 4.3 and Theorem 4.4 we may easily obtain the validity of Theorem 3.4 for general $\Delta_{U,W}$-integrable functions.

5 Properties of the integral

For the sake of completeness, here we give a more precise proof of Theorem 4.1 from [17] than it is given therein and also a detail proof of Theorem 4.4 therein.

Theorem 5.1 *Let* $\mathbf{h}, \mathbf{g} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ *and* $E \in \Sigma$*. If* $\mathbf{h} + \mathbf{g} = 0$*, then* $\int_E \mathbf{h}\,\mathrm{d}\mathbf{m} + \int_E \mathbf{g}\,\mathrm{d}\mathbf{m} = 0$.

Proof. Let (U_1, W_1), (U_2, W_2), $(U_{\mathbf{h}}, W_{\mathbf{h}})$, $(U_{\mathbf{g}}, W_{\mathbf{g}}) \in \mathcal{U} \times \mathcal{W}$. We have to show that if $\mathbf{h}_n, \mathbf{g}_n$, $n \in \mathbb{N}$ are sequences of Δ_{U_1,W_1}-, Δ_{U_2,W_2}-simple functions U_1-, U_2-converging on $T \setminus N_{U_{\mathbf{h}},W_{\mathbf{h}}}$, $T \setminus N_{U_{\mathbf{g}},W_{\mathbf{g}}}$, where $N_{U_{\mathbf{h}},W_{\mathbf{h}}} \in \mathcal{O}(\hat{\mathbf{m}}_{U_{\mathbf{h}},W_{\mathbf{h}}})$, $N_{U_{\mathbf{g}},W_{\mathbf{g}}} \in \mathcal{O}(\hat{\mathbf{m}}_{U_{\mathbf{g}},W_{\mathbf{g}}})$, to the functions $\mathbf{h}, \mathbf{g}$, such that $\mathbf{h} + \mathbf{g} = 0$ and the integrals $\int_\cdot \mathbf{h}_n \mathrm{d}\mathbf{m}$, $\int_\cdot \mathbf{g}_n \mathrm{d}\mathbf{m}$, are uniformly (W_1, σ)- and (W_2, σ)-additive measures on Σ, respectively, then the sequence $\mathbf{k}_n = \mathbf{h}_n + \mathbf{g}_n$, $n \in \mathbb{N}$ of functions $(U_1 \vee U_2)$-converges $\hat{\mathbf{m}}_{U_{\mathbf{h}} \wedge U_{\mathbf{g}}, W_{\mathbf{h}} \vee W_{\mathbf{g}}}$-a.e. to the function $\mathbf{k} = \mathbf{h} + \mathbf{g}$.

Indeed, let $\varepsilon > 0$ be chosen arbitrarily. Denote by $U = U_1 \vee U_2$, $N = N_{U_{\mathbf{h}},W_{\mathbf{h}}} \cup N_{U_{\mathbf{g}},W_{\mathbf{g}}}$ and put

$$n_0 = \max\{n \in \mathbb{N}; \min\{p_{U_k}(\mathbf{h}_n(t) - \mathbf{h}(t)) < \varepsilon\}; k = 1, 2\}.$$

Then for every $t \in T \setminus N$ and $n \geq n_0$, $n \in \mathbb{N}$, we have

$$\begin{aligned} p_U(\mathbf{h}_n(t) + \mathbf{g}_n(t)) &= p_U(\mathbf{h}_n(t) - \mathbf{h}(t) + \mathbf{g}_n(t) - \mathbf{g}(t) + \mathbf{h}(t) + \mathbf{g}(t)) \\ &\leq p_U(\mathbf{h}_n(t) - \mathbf{h}(t)) + p_U(\mathbf{g}_n(t) - \mathbf{g}(t)) + p_U(\mathbf{h}(t) + \mathbf{g}(t)) \\ &\leq p_{U_1}(\mathbf{h}_n(t) - \mathbf{h}(t)) + p_{U_2}(\mathbf{g}_n(t) - \mathbf{g}(t)) < 2\varepsilon. \end{aligned}$$

Since the measure $\mathbf{m}$ is of σ-finite $(\mathcal{U}, \mathcal{W})$-semivariation, there exists a map $\varphi_1 : U \mapsto \varphi_1(U)$ with $(U, \varphi_1(U)) \in \mathcal{U} \times \mathcal{W}$, such that $\mathbf{m}$ is of σ_{φ_1}-finite $(U, \varphi_1(U))$-semivariation. Then by Lemma 2.7 the measure $\mathbf{m}$ is of σ_φ-finite $(U, \varphi(U))$-semivariation, where $\varphi(U) = \varphi_1(U) \vee W_1 \vee W_2$. Denote by $W = \varphi(U)$. Then there exist pairwise disjoint sets $A_j \in \Delta_{U,W}$, $j \in \mathbb{N}$, such that $\bigcup_{j=1}^\infty A_j = T$.

Let $E = \bigcup_{i=1}^\infty E_i$, $E_i \cap E_j = \emptyset$, $E_i, E_j \in \Sigma$, for $i \neq j$, $i, j \in \mathbb{N}$. Then

$$\begin{aligned} p_W\left(\int_{\bigcup_{i=I+1}^\infty E_i} \mathbf{k}_n \,\mathrm{d}\mathbf{m}\right) &\leq p_W\left(\int_{\bigcup_{i=I+1}^\infty E_i} \mathbf{h}_n \,\mathrm{d}\mathbf{m}\right) + p_W\left(\int_{\bigcup_{i=I+1}^\infty E_i} \mathbf{g}_n \,\mathrm{d}\mathbf{m}\right) \\ &\leq p_{W_1}\left(\int_{\bigcup_{i=I+1}^\infty E_i} \mathbf{h}_n \,\mathrm{d}\mathbf{m}\right) + p_{W_2}\left(\int_{\bigcup_{i=I+1}^\infty E_i} \mathbf{g}_n \,\mathrm{d}\mathbf{m}\right) \end{aligned}$$

where $I, n \in \mathbb{N}$. The above inequality implies that the integrals $\int_\cdot \mathbf{k}_n \,\mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly (W, σ)-additive measures on Σ. Thus, for every $\varepsilon > 0$ there exists $J \in \mathbb{N}$, such that

$$p_W\left(\int_{E \cap \bigcup_{j=J+1}^\infty A_j} \mathbf{k}_n \mathrm{d}\mathbf{m}\right) \leq \varepsilon$$

uniformly for every $n \in \mathbb{N}$. Denote by $F_{j,k} \in \Delta_{U,W}$, $F_{j,k} \in A_j$, $F_{j,k} \subset F_{j,k+1}$, for $j = 1, 2, \ldots, J$, $k \in \mathbb{N}$, a nondecreasing sequence of sets, such that the sequence $\mathbf{k}_n$, $n \in \mathbb{N}$, of functions uniformly $(U, F_{j,k})$-converges to $\mathbf{k}$.

Then there exists $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$, $n \in \mathbb{N}$, and $p \in \mathbb{N}$ there holds

$$\|\mathbf{k}_n - \mathbf{k}_{n+p}\|_{F_{j,k},U} < \frac{\varepsilon}{3 \cdot \sum_{j=1}^{J} \hat{\mathbf{m}}_{U,W}(A_j)},$$

for $j = 1, 2, \ldots, J$, $k \in \mathbb{N}$.

Choose the sets $F_{j,k}$, $k \in \mathbb{N}$, such that for every $p \in \mathbb{N}$,

$$\left| \int_{\cdot} \mathbf{k}_{n+p} \mathrm{d}\mathbf{m} \right|_W (A_j \setminus F_{j,k} \setminus N) < \frac{\varepsilon}{3 \cdot 2^j}, \quad j = 1, 2, \ldots, J.$$

Repeating now the proof of Theorem 3.8 in [17] from the inequality (5) on, replacing formally here $\mathbf{f}_n$ by $\mathbf{k}_n$, $n \geq n_0$, we obtain that

$$p_W \left(\int_E \mathbf{k}_n \mathrm{d}\mathbf{m} - \int_E \mathbf{k}_{n+p} \mathrm{d}\mathbf{m} \right) < 3\varepsilon.$$

The uniqueness of the integral is proved. □

Theorem 5.2 *(a) The family $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ is a vector space.*

(b) For every $E \in \Sigma$, the map $\int_E (\cdot) \,\mathrm{d}\mathbf{m} : \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta} \to \mathbf{Y}$ is a linear operator.

Proof. Let $\lambda \in \mathbb{K}$ and $\mathbf{g}, \mathbf{h} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$. We have to show that

(a) $\lambda \mathbf{g} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ and $\mathbf{g} + \mathbf{h} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$, for all $\lambda \in \mathbb{K}$; and

(b) for every $E \in \Sigma$ there holds: $\int_E (\lambda \mathbf{g}) \,\mathrm{d}\mathbf{m} = \lambda \int_E \mathbf{g} \,\mathrm{d}\mathbf{m}$ and $\int_E (\mathbf{g} + \mathbf{h}) \,\mathrm{d}\mathbf{m} = \int_E \mathbf{g} \,\mathrm{d}\mathbf{m} + \int_E \mathbf{h} \,\mathrm{d}\mathbf{m}$.

If $\lambda = 0$ and $\mathbf{g} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$, then $0 \cdot \mathbf{g} = 0 \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ and

$$\int_E (0 \cdot \mathbf{g}) \,\mathrm{d}\mathbf{m} = 0 = 0 \cdot \int_E \mathbf{g} \,\mathrm{d}\mathbf{m}.$$

Let $\lambda \neq 0$ and $\mathbf{g} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$. Then there exists a couple $(U_{\mathbf{g}}, W_{\mathbf{g}}) \in \mathcal{U} \times \mathcal{W}$, and a sequence $\mathbf{g} \in \mathcal{S}_{U_{\mathbf{g}},W_{\mathbf{g}}}$, $n \in \mathbb{N}$, such that

(i) $\lim_{n\to\infty} p_{U_{\mathbf{g}}}(\mathbf{g}_n(t) - \mathbf{g}(t)) = 0$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e., and

(ii) $\lim_{n\to\infty} p_{W_{\mathbf{g}}} \left(\int_E \mathbf{g}_n \,\mathrm{d}\mathbf{m} - \int_E \mathbf{g} \,\mathrm{d}\mathbf{m} \right) = 0$, for $E \in \Sigma$.

We have

$$\begin{aligned} p_{W_{\mathbf{g}}} \left(\lambda \int_E \mathbf{g} \,\mathrm{d}\mathbf{m} - \int_E (\lambda \mathbf{g}_n) \,\mathrm{d}\mathbf{m} \right) &\leq p_{W_{\mathbf{g}}} \left(\lambda \int_E \mathbf{g} \,\mathrm{d}\mathbf{m} - \lambda \int_E \mathbf{g}_n \,\mathrm{d}\mathbf{m} \right) \\ &+ p_{W_{\mathbf{g}}} \left(\lambda \int_E \mathbf{g}_n \,\mathrm{d}\mathbf{m} - \int_E (\lambda \mathbf{g}_n) \,\mathrm{d}\mathbf{m} \right) \\ &\leq |\lambda| \cdot \varepsilon. \end{aligned} \tag{5}$$

So, the sequence $\lambda\mathbf{g}_n$, $n \in \mathbb{N}$ satisfies the definition of the integrable function for the function $\lambda\mathbf{g}$ and $\lambda\mathbf{g} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$. By (5) we get

$$\begin{aligned}
&p_{W_{\mathbf{g}}}\left(\lambda \int_E \mathbf{g}\,\mathrm{d}\mathbf{m} - \int_E (\lambda\mathbf{g}_n)\,\mathrm{d}\mathbf{m}\right) \\
&= p_{W_{\mathbf{g}}}\left(\lambda \int_E \mathbf{g}\,\mathrm{d}\mathbf{m} - \int_E (\lambda\mathbf{g}_n)\,\mathrm{d}\mathbf{m}\right) + p_{W_{\mathbf{g}}}\left(\lambda \int_E \mathbf{g}\,\mathrm{d}\mathbf{m} - \int_E (\lambda\mathbf{g})\,\mathrm{d}\mathbf{m}\right) \\
&\le p_{W_{\mathbf{g}}}\left(\lambda \int_E \mathbf{g}\,\mathrm{d}\mathbf{m} - \int_E (\lambda\mathbf{g}_n)\,\mathrm{d}\mathbf{m}\right) + p_{W_{\mathbf{g}}}\left(\int_E (\lambda\mathbf{g}_n)\,\mathrm{d}\mathbf{m} - \int_E (\lambda\mathbf{g})\,\mathrm{d}\mathbf{m}\right) \\
&\le |\lambda| \cdot \varepsilon + p_{W_{\mathbf{g}}}\left(\lambda \int_E \mathbf{g}_n\,\mathrm{d}\mathbf{m} - \int_E (\lambda\mathbf{g})\,\mathrm{d}\mathbf{m}\right) \\
&\le 2|\lambda| \cdot \varepsilon.
\end{aligned}$$

Thus,

$$\lambda \int_E \mathbf{g}\,\mathrm{d}\mathbf{m} = \int_E (\lambda\mathbf{g})\,\mathrm{d}\mathbf{m}.$$

Similarly, by definition of the integrable function, there exists a couple $(U_{\mathbf{h}}, W_{\mathbf{h}}) \in \mathcal{U} \times \mathcal{W}$, and a sequence $\mathbf{h}_n \in \mathcal{S}_{U_{\mathbf{h}},W_{\mathbf{h}}}$, $n \in \mathbb{N}$, such that

(iii) $\lim_{n\to\infty} p_{U_{\mathbf{h}}}(\mathbf{h}_n(t) - \mathbf{h}(t)) = 0$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e., and

(iv) $\lim_{n\to\infty} p_{W_{\mathbf{h}}}\left(\int_E \mathbf{h}_n\,\mathrm{d}\mathbf{m} - \int_E \mathbf{h}\,\mathrm{d}\mathbf{m}\right) = 0$, for $E \in \Sigma$.

Then we have

$$\begin{aligned}
&p_{W_{\mathbf{g}}\vee W_{\mathbf{h}}}\left(\left[\int_E \mathbf{g}\,\mathrm{d}\mathbf{m} + \int_E \mathbf{h}\,\mathrm{d}\mathbf{m}\right] - \int_E (\mathbf{g}_n + \mathbf{h}_n)\,\mathrm{d}\mathbf{m}\right) \\
&\le p_{W_{\mathbf{g}}\vee W_{\mathbf{h}}}\left(\int_E \mathbf{g}\,\mathrm{d}\mathbf{m} - \int_E \mathbf{g}_n\,\mathrm{d}\mathbf{m}\right) + p_{W_{\mathbf{g}}\vee W_{\mathbf{h}}}\left(\int_E \mathbf{h}\,\mathrm{d}\mathbf{m} - \int_E \mathbf{h}_n\,\mathrm{d}\mathbf{m}\right).
\end{aligned}$$

So, the sequence $(\mathbf{g}_n - \mathbf{h}_n)$ satisfies the definition of the integrable function for the function $(\mathbf{g} - \mathbf{h})$. The proof is complete. □

6 An example of the integral in C. B. L. C. S.

Put $\mathbf{X} = \mathcal{D}$, where $\mathcal{D}$ denotes the space of all infinitely many times differentiable real functions with compact supports on the real line. It is well-known, cf. [23], that $\mathcal{D} = \mathrm{injlim}_{n\to\infty}\mathcal{D}_{[-n,+n]}$, where $\mathcal{D}_{[-n,+n]} = \mathrm{injlim}_{m\to\infty}\mathcal{D}^m_{[-n,+n]}$, $n \in \mathbb{N}$, are Fréchet spaces for every $n \in \mathbb{N}$, and $\mathcal{D}^m_{[-n,+n]}$, $n, m \in \mathbb{N}$, is a Banach space equipped with the norm $p_{n,m}(\mathbf{x}) = \sup|\mathbf{x}^{(k)}(\xi)|$, where the supremum is taken over all $\xi \in [-n,+n]$, $0 \le k \le m$, and $\mathbf{x}^{(k)}(\xi)$ denotes the k-th derivative of the function $\mathbf{x} \in \mathcal{D}^m_{[-n,+n]}$ at the point $\xi \in [-n,+n]$, $0 \le k \le m$. The space $\mathcal{D}$ is a (von Neumann) bornological locally convex topological non-metrizable vector space, cf. Remark 2.1. For further reading about $\mathcal{D}$, cf. [23].

Denote by $\mathcal{U}$ the set of all $U \subset \mathcal{D}$, such that

$$U = \left\{\sum_{i=1}^{I} \alpha_i\mathbf{e}_i;\ \sum_{i=1}^{I} |\alpha_i| \le 1,\ \mathbf{e}_i \in \mathcal{D},\ \alpha_i \in \mathbb{R},\ i = 1, 2, \ldots, I\right\},$$

where $\mathbf{e}_i \in \mathcal{D}$, $i = 1, 2, \ldots, I$, are linearly independent functions over $\mathbb{R}$ and

$$\mathbf{e}_1(t) = \begin{cases} e^{-\frac{1}{(t-1)^2}} & \text{if } t \in [-1, 1), \\ 0 & \text{if } t \notin [-1, 1). \end{cases}$$

Let $U_0 = \{\alpha\, \mathbf{e}_1; |\alpha| \leq 1\}$. The family of all sets $U \in \mathcal{U}$ forms a Banach disk basis of a bornology on $\mathbf{X}$. Denote by p_U the Minkowski functional of the set U and $\mathbf{X}_U$ the linear envelope of the set $U \in \mathcal{U}$. Clearly, $\mathbf{X}_U$ is a Banach space equipped with the norm p_U. The inductive limit

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U$$

is an example of a set $\mathcal{D}$ equipped with the bornological convergence (which differs from the usual convergence in $\mathcal{D}$ with respect to the metric given by the system of seminorms $p_{n,m}$, $n, m \in \mathbb{N}$), cf. [16].

Measure Put $\mathbf{X} = \mathbf{Y}$ and $\mathcal{U} = \mathcal{W}$. Consider a measure $\mathbf{m}(E)\mathbf{x} = L(\mathbf{x}) \cdot \lambda(E)$, where $\mathbf{x} \in \mathbf{X}$ and λ is the Lebesgue measure on the real line, $E \in \Sigma$ is a Lebesgue measurable set and $L \in L(\mathbf{X}, \mathbf{Y})$ is a continuous linear operator. So, for every $U \in \mathcal{U}$ there exists an element $W \in \mathcal{W}$, such that $L(U) \subset W$. Denote this function by $\varphi : \mathcal{U} \to \mathcal{W}$. Take a couple $(U, W) \in \mathcal{U} \times \mathcal{W}$, and $\varphi : U \mapsto W$. Clearly, $L(U_0) = W_0 \subset L(U) \subset \varphi(U) = W$ for every $U \in \mathcal{U}$.

The set function $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$, $\mathbf{x} \in \mathbf{X}_U$, $A \in \Sigma$, is a σ-additive measure in the strong operator topology of the space $L(\mathbf{X}_U, \mathbf{Y}_W)$. Indeed, let $E = \bigcup_{i=1}^{\infty} E_i$, where $E_i \cap E_j = \emptyset$, $E_i, E_j \in \Sigma$, for $i, j \in \mathbb{N}$, $i \neq j$. Then

$$\begin{aligned} \mathbf{m}\left(A \cap \bigcup_{i=1}^{\infty} E_i\right)\mathbf{x} &= \lambda\left(\bigcup_{i=1}^{\infty}(A \cap E_i)\right) L(\mathbf{x}) = L(\mathbf{x}) \sum_{i=1}^{\infty} \lambda(A \cap E_i) \\ &= \sum_{i=1}^{\infty} \lambda(A \cap E_i) L(\mathbf{x}) = \sum_{i=1}^{\infty} \mathbf{m}(A \cap E_i)\mathbf{x} \in \mathbf{Y}_W, \end{aligned}$$

for $\mathbf{x} \in \mathbf{X}_U$.

Null sets Let $E \in \Sigma$ be of finite (U, W)-semivariation (see (6)). Since $L(U) \subset W$ and U is a convex set, then the (U, W)-semivariation

$$\begin{aligned} \hat{\mathbf{m}}_{U,W}(E) &= \sup p_W \left(\sum_{j=1}^{J} \mathbf{m}(E \cap E_j)\mathbf{x}_j\right) \\ &= \sup p_W \left(\sum_{j=1}^{J} \lambda(E \cap E_j) L(\mathbf{x}_j)\right) \\ &= \sup p_W \left(L\left\{\sum_{j=1}^{J} \frac{\lambda(E \cap E_j)}{\lambda(E)} \mathbf{x}_j\right\}\right) \cdot \lambda(E) \\ &\leq \lambda(E), \end{aligned} \tag{6}$$

where the supremum is taken over all disjoint $E_j \in \Sigma$ and $\mathbf{x}_j \in U$, $j = 1, 2, \ldots, J$. We see that the σ-ideal $\mathcal{N}(\hat{\mathbf{m}}_{U,W})$ of $\hat{\mathbf{m}}_{U,W}$-null sets is dominated by the σ-ideal of λ-null sets, i.e. $\lambda(E) = 0$, $E \in \Sigma_{U,W}$ (= the family of all sets in Σ of finite (U, W)-semivariation), $(U, W) \in \mathcal{U} \times \mathcal{W}$, $W = \varphi(U) \Rightarrow \hat{\mathbf{m}}_{U,W}(E) = 0$.

Simple functions Denote by $\mathbf{X}i$ the family of all functions in $\mathcal{D}$ of the type $\mathbf{x} = \sum_{i=1}^{I} c_i \mathbf{e}_i$, where $c_i = \mathrm{const}(t) \in \mathbb{R}$, $\mathbf{e}_i \in \mathbf{X}_U$, $U \in \mathcal{U}$, for $i = 1, 2, \ldots, I$.

Denote by $\mathcal{S}_{U,W}$ the family of all functions of the type

$$\mathbf{f}(t) = \sum_{j=1}^{J} \mathbf{x}_j \chi_{E_j} = \sum_{j=1}^{J} \left(\sum_{i=1}^{I} c_{i,j} \mathbf{e}_i \right) \chi_{E_j}(t) = \sum_{i=1}^{I} \alpha_i(t) \mathbf{e}_i,$$

where

$$\alpha_i = \sum_{j=1}^{J} c_{i,j} \chi_{E_j}, \tag{7}$$

for $E_j \in \Sigma_{U,W}$, $\mathbf{x}_j \in \mathbf{X}i$, $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J$. We say that $\mathcal{S}_{U,W}$ is a space of $\Sigma_{U,W}$-simple functions. Then the integral of $\mathbf{f} \in \mathcal{S}_{U,W}$ is given by

$$\begin{aligned}
\mathbf{y}_E &= \int_E \mathbf{f}\, \mathrm{d}\mathbf{m} = \sum_{j=1}^{J} \mathbf{m}(E_j \cap E) \mathbf{x}_j = \sum_{j=1}^{J} \lambda(E_j \cap E) L(\mathbf{x}_j) \\
&= \sum_{j=1}^{J} \lambda(E_j \cap E) L \left(\sum_{i=1}^{I} c_{i,j} \mathbf{e}_i \right) = \sum_{i=1}^{I} \left[\sum_{j=1}^{J} \lambda(E_j \cap E) c_{i,j} \right] L(\mathbf{e}_i) \\
&= \sum_{i=1}^{I} \beta_i L(\mathbf{e}_i),
\end{aligned}$$

where $\beta_i = \sum_{j=1}^{J} \lambda(E_j \cap E) c_{i,j}$, $i = 1, 2, \ldots, I$, $E \in \Sigma$. Observe that $\mathbf{X}i \subset \mathcal{D}$, and $\int_E \mathbf{f}\, \mathrm{d}\mathbf{m} \in \mathcal{D}$, $E \in \Sigma$, for every $\mathbf{f} \in \mathcal{S}_{U,W}$.

Integral Let for each $i = 1, 2, \ldots, I$, a sequence of functions $\alpha_{i,K} : \mathbb{R} \to \mathbb{R}$, $K \in \mathbb{N}$, being of the type (7), converges λ-a.e. to a function $\alpha_{i,\infty} : \mathbb{R} \to \mathbb{R}$, and is dominated by a Lebesgue integrable function (so, $\alpha_{i,\infty}$ is a Lebesgue integrable function), i.e. $\alpha_{i,K} = \sum_{j_k=1}^{J_K} c_{i,j_k} \chi_{E_{j_k},i}$, where $E_{j_k,i} \in \Sigma_{U,W}$, for $i = 1, 2, \ldots, I$, $\lim_{K\to\infty} J_K = \infty$, and $j_k = 1, 2, \ldots, J_K$. Then $\mathbf{f}_K = \sum_{i=1}^{I} \alpha_{i,K} \mathbf{e}_i$, $K \in \mathbb{N}$, is a sequence of $\Sigma_{U,W}$-simple functions converging λ-a.e. to the (Dobrakov) $\Sigma_{U,W}$-integrable function $\mathbf{f} = \sum_{i=1}^{I} \alpha_{i,\infty} \mathbf{e}_i$ and the integral may be computed easily as follows:

$$\int_E \mathbf{f}\, \mathrm{d}\mathbf{m} = \sum_{i=1}^{I} \left(\lim_{K\to\infty} \int_E \alpha_{i,K}\, \mathrm{d}\lambda \right) L(\mathbf{e}_i) = \sum_{i=1}^{I} \beta_{i,\infty} L(\mathbf{e}_i),$$

where $\beta_{i,\infty} = \lim_{K\to\infty} \beta_{i,K}$.

The described integral is linear, absolutely continuous, and all theorems of the Dobrakov's integration theory can be applied. However, the choice of the Banach spaces $\mathbf{X}_U, \mathbf{Y}_W$ depends on the function which we integrate. The simplicity of this example consists from the fact that the considered operator-valued measure is dominated by a non-negative real measure. It has implied the common σ-algebra for every integrable function (the domain of its integral). Considering the general case, we need an additional assumption concerning a notion of σ-finiteness of the measure to construct an integral.

References

[1] Bartle, R. G.: A general bilinear vector integral. *Studia Math.* **15** (1956), 337–352.

[2] Boccuto, A. – Haluška, J. – Hutník, O.: On product measures in complete bornological locally convex spaces. *Atti Sem. Mat. Fis. Univ. Modena* (submitted).

[3] Bombal, F.: Medida e integration bornologica, (in Spanish). *Rev. R. Acad. Ci. Madrid* **75** (1981), 115–137.

[4] Bravo, R. – Jiménez Guerra, P.: Linear operators and vector integrals. *Math. Japonica* **36** (1991), 255–262.

[5] Ballvé, M. E.: L^α spaces for operator valued measures. *Ann. Sci. Math. Québec* **14** (1990), 9–15.

[6] Ballvé, M. E.: Geometry and integration for operator valued measures. *J. Math. Anal. Appl.* **165** (1992), 62–70.

[7] Debieve, C.: Integration of vector valued functions with respect to vector valued measures. *Rev. Roumaine Math. Pures Appl.* **26** (1981), 943–957.

[8] Diestel, J. – Uhl, J. J.: *Vector Measures*. Amer. Math. Soc., Providence, 1977.

[9] Dobrakov, I.: On integration in Banach spaces, I. *Czechoslovak Math. J.* **20** (1970), 511–536.

[10] Dobrakov I.: On integration in Banach spaces, II. *Czechoslovak Math. J.* **20** (1970), 680–695.

[11] Dunford, N. – Schwartz, J. T.: *Linear Operators. Part I: General Theory*. Interscience Publisher, New York, 1958.

[12] Halmos, P. P.: *Measure Theory*. Springer, New York, 1950.

[13] Haluška, J.: On the continuity of the semivariation in locally convex spaces. *Acta Univer. Carolin. - Math. Phys.* **32** (1991), 23–28.

[14] Haluška, J.: On convergences of functions in complete bornological locally convex spaces. *Rev. Roumaine Math. Pures Appl.* **38** (1993), 327–337.

[15] Haluška, J.: On a lattice structure of operator spaces in complete bornological locally convex spaces. *Tatra Mt. Math. Publ.* **2** (1993), 143–147.

[16] Haluška, J.: On lattices of set functions in complete bornological locally convex spaces. *Simon Stevin* **67** (1993), 27–48.

[17] Haluška, J.: On integration in complete bornological locally convex spaces. *Czechoslovak Math. J.* **47** (1997), 205–219.

[18] Haluška, J.–Hutník, O.: On vector integral inequalities (manuscript).

[19] Hogbe-Nlend, H.: *Bornologies and Functional Analysis*. North-Holland, Amsterdam–New York–Oxford, 1977.

[20] Jarchow, H.: *Locally Convex Spaces*. Teubner, Stuttgart, 1981.

[21] Radyno, J. V.: *Linear Equations and the Bornology* (in Russian). Izd. Bel. Gosud. Univ., Minsk, 1982.

[22] Rao Chivukula, R. – Sastry, A. S.: Product vector measures via Bartle integrals. *J. Math. Anal. Appl.* **96** (1983), 180–195.

[23] Robertson, A. P. – Robertson, W. J.: *Topological Vector Spaces*. Univ. Press, Cambridge, 1964.

[24] Roy, S. K. – Charkaborty, N. D.: Integration of vector valued functions with respect to an operator valued measure. *Czechoslovak Math. J.* **36** (1986), 198–209.

[25] Smith, W. – Tucker, D. H.: Weak integral convergence theorem and operator measures. *Pacific J. Math.* **111** (1984), 243–256.

[26] Wong, N.-Ch.: The triangle of operators, topologies and bornologies. In: *ArXiv Mathematics*, e-print arXiv:math/0506183 (2005).

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