Mediterr. J. Math. 7 (2010), 353–372
DOI 10.1007/s00009-010-0048-x
1660-5446/10/030353-20, *published online* April 22, 2010

Mediterranean Journal of Mathematics

On Domination and Bornological Product Measures

Ján Haluška and Ondrej Hutník*

Abstract. The bornological product measures via the generalized Dobrakov integral in complete bornological locally convex spaces are studied using the domination of considered vector measures. A Fubini-type theorem for such product measures is proven.

Mathematics Subject Classification (2010). Primary 46G10; Secondary 28B05.

Keywords. Bilinear integral, bornology, locally convex topological vector space, Dobrakov integral, bornological product measure, dominated vector measure, Fubini theorem.

1. Introduction and preliminaries

The problem of product of vector measures has been studied in several papers, where some conditions for the existence of the product of vector measures have been given, see e.g. [12] and [27] for further references. In [10] the problem has been solved in connection with the bilinear integral of Bartle, cf. [1] and domination of vector measures (the domination is understood in the sense to find a non-negative finite measure with respect to which the given one is absolutely continuous, cf. [26]). Product of vector measures via the Bartle integral has also been investigated in [25].

The integration technique developed by the first author in [17] for the complete bornological locally convex vector spaces (C. B. L. C. S., for short) completely generalizes the Dobrakov integral, cf. [4], [5], to non-metrizable vector spaces and provides a good tool to study bornological product measures. Note here the paper of Ballvé and Jiménez Guerra, cf. [2], where we can find a list of reference papers to the problem on bornological product measures. In [20] the bornological product measures in connection with the

This paper was supported by Grants VEGA 2/0097/08 and CNR–SAS project 2007–2009 “Integration in abstract structures”.
*Corresponding author.

above mentioned generalization of Dobrakov integral is studied and a Fubini-type theorem for them is proved. The general Fubini theorem for bornological product measures is proven in [21].

In this paper we show the applicability of our integral to bornological product measures. In Section 2 we recall the definition of bornological product measure and state some sufficient conditions for its existence. The domination of operator-valued measures is discussed in Section 4 where the question whether the bornological product measure of dominated measures is also dominated is solved. In Section 5 the Fubini-type theorem for dominated bornological product measures is established.

1.1. C. B. L. C. S.

In the following we recall basic facts and necessary notions from the integration theory in C. B. L. C. S., cf. [17]. The detailed description of the theory of C. B. L. C. S. may be found in [22], [23] and [24].

Let $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ be Hausdorff C. B. L. C. S. over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex numbers $\mathbb{C}$, equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}$, $\mathfrak{B}_{\mathbf{Y}}$, $\mathfrak{B}_{\mathbf{Z}}$, respectively. We say that the basis $\mathcal{U}$ of bornology $\mathfrak{B}_{\mathbf{X}}$ has the *vacuum vector* [1] $U_0 \in \mathcal{U}$, if $U_0 \subset U$ for every $U \in \mathcal{U}$. Let the bases $\mathcal{U}$, $\mathcal{W}$, $\mathcal{V}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-, $\mathfrak{B}_{\mathbf{Z}}$-bounded Banach disks in $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ with vacuum vectors $U_0 \in \mathcal{U}$, $U_0 \neq \{0\}$, $W_0 \in \mathcal{W}$, $W_0 \neq \{0\}$, $V_0 \in \mathcal{V}$, $V_0 \neq \{0\}$, respectively. Recall that *a (separable) Banach disk* in $\mathbf{X}$ is a set $U \in \mathfrak{B}_{\mathbf{X}}$ which is closed, absolutely convex and the linear span $\mathbf{X}_U$ of which is a (separable) Banach space. So, the space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_U$, $U \in \mathcal{U}$, i.e.

$$\mathbf{X} = \operatorname*{injlim}_{U \in \mathcal{U}} \mathbf{X}_U,$$

cf. [23], where $\mathcal{U}$ is directed by inclusion (analogously for $\mathbf{Y}$ and $\mathcal{W}$, $\mathbf{Z}$ and $\mathcal{V}$, respectively).

Since $\mathbf{X}_U$, $U \in \mathcal{U}$, in the definition of C. B. L. C. S. is a Banach space, it is enough to deal with sequences instead of nets and therefore we introduce the following bornological convergence in the sense of Mackey. We say that a sequence $(\mathbf{x}_n)_{n=1}^{\infty}$ of elements from $\mathbf{X}$ converges bornologically with respect to the bornology $\mathfrak{B}_{\mathbf{X}}$ with the basis $\mathcal{U}$ to $\mathbf{x} \in \mathbf{X}$, shortly *$\mathcal{U}$-converges*, if there exists $U \in \mathcal{U}$, such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that $(\mathbf{x}_n - \mathbf{x}) \in \varepsilon U$ for every $n \geq n_0$. We write $\mathbf{x} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{x}_n$. To be more precise, we will sometimes call this the U-convergence of elements from $\mathbf{X}$ to show explicitly which $U \in \mathcal{U}$ we have in the mind.

Remark 1.1. A classical bornology consists of all sets which are bounded in the von Neumann sense, i.e. for a locally convex topological vector space $\mathbf{X}$ equipped with a family of seminorms Q, the set B is *bounded* (or belongs to the von Neumann bornology) if and only if for every $q \in Q$ there exists a constant C_q such that $q(\mathbf{x}) \leq C_q$ for every $\mathbf{x} \in B$.

[1] in literature we can find also as terms as the *ground state* or *marked element* or *fiducial vector* or *mother wavelet* depending on the context

1.2. Operator spaces

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $U_1, U_2 \in \mathcal{U}$ we have: $U_1 \wedge U_2 = U_1 \cap U_2$, and $U_1 \vee U_2 = \operatorname{acs}(U_1 \cup U_2)$, where acs denotes the topological closure of the absolutely convex span of the set; analogously for $\mathcal{W}$ and $\mathcal{V}$. For $(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U}\times\mathcal{W}\times\mathcal{V}$, we write $(U_1, W_1, V_1) \ll (U_2, W_2, V_2)$ if and only if $U_1 \subset U_2$, $W_1 \supset W_2$, and $V_1 \supset V_2$.

We use Φ, Ψ, Γ to denote the *classes of all functions* $\mathcal{U} \to \mathcal{W}$, $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$ *with orders* $<_\Phi$, $<_\Psi$, $<_\Gamma$ defined as follows: for $\varphi_1, \varphi_2 \in \Phi$ we write $\varphi_1 <_\Phi \varphi_2$ whenever $\varphi_1(U) \subset \varphi_2(U)$ for every $U \in \mathcal{U}$ (analogously for $<_\Psi$, $<_\Gamma$ and $\mathcal{W} \to \mathcal{V}$, $\mathcal{U} \to \mathcal{V}$, respectively).

Denote by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. We suppose $L(\mathbf{X}, \mathbf{Y}) \subset \Phi$. Analogously, $L(\mathbf{Y}, \mathbf{Z}) \subset \Psi$ and $L(\mathbf{X}, \mathbf{Z}) \subset \Gamma$. The bornologies $\mathfrak{B}_\mathbf{X}$, $\mathfrak{B}_\mathbf{Y}$, $\mathfrak{B}_\mathbf{Z}$ are supposed to be stronger than the corresponding von Neumann bornologies, i.e. the vector operations on the spaces $L(\mathbf{X}, \mathbf{Y})$, $L(\mathbf{Y}, \mathbf{Z})$, $L(\mathbf{X}, \mathbf{Z})$ are compatible with the topologies, and the bornological convergence implies the topological convergence. Note that in the terminology [24] the space $L(\mathbf{X}, \mathbf{Y})$ (as an inductive limit of seminormed spaces) is a bornological convex vector space. For a more detailed explanation of the topological and bornological methods of functional analysis in connection with operators, cf. [28].

1.3. Set functions

Let S and T be two non-void sets. Let Δ and ∇ be two δ-rings of subsets of sets S and T, respectively. If $\mathcal{A}$ is a system of subsets of the set S, then $\sigma(\mathcal{A})$ (resp. $\delta(\mathcal{A})$) denotes the σ-ring (resp. δ-ring) generated by the system $\mathcal{A}$. Put $\Sigma = \sigma(\Delta)$ and $\Xi = \sigma(\nabla)$. We use χ_E to denote the characteristic function of the set E. By $p_U : \mathbf{X} \to [0, +\infty]$ we denote the Minkowski functional of the set $U \in \mathcal{U}$, i.e. $p_U(\mathbf{x}) = \inf_{\mathbf{x}\in\lambda U} |\lambda|$ (if U does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_U(\mathbf{x}) = +\infty$). Similarly, p_W and p_V indicate the Minkowski functionals of the sets $W \in \mathcal{W}$ and $V \in \mathcal{V}$, respectively.

For every $(U, W) \in \mathcal{U} \times \mathcal{W}$, denote by $\hat{\mathbf{m}}_{U,W} : \Sigma \to [0, +\infty]$ *a* (U, W)-*semivariation* of a charge (= finitely additive measure) $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ given by

$$\hat{\mathbf{m}}_{U,W}(E) = \sup p_W \left(\sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i \right), \quad E \in \Sigma,$$

where the supremum is taken over all finite sets $\{\mathbf{x}_i \in U, i = 1, 2, \ldots, I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1, 2, \ldots, I\}$. For $\{E_i \in \Delta, i = 1, 2, \ldots, I\}$ by Corollary 3 to Proposition 9, § 1 in [3] we get $E \cap E_i \in \Delta$ for $E \in \Sigma$, and hence $\hat{\mathbf{m}}_{U,W}(E)$ is well defined. Note that this result does not hold if Σ is the σ-algebra generated by Δ. It is well-known that $\hat{\mathbf{m}}_{U,W}$ is a submeasure, i.e. a monotone, subadditive set function, and $\hat{\mathbf{m}}_{U,W}(\emptyset) = 0$. The family $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}} = \{\hat{\mathbf{m}}_{U,W};\ (U, W) \in \mathcal{U} \times \mathcal{W}\}$ is said to be the $(\mathcal{U}, \mathcal{W})$-semivariation of $\mathbf{m}$.

For every $(U,W) \in \mathcal{U} \times \mathcal{W}$, denote by $\|\mathbf{m}\|_{U,W} : \Sigma \to [0,+\infty]$ *a scalar (U,W)-semivariation* of a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ defined as

$$\|\mathbf{m}\|_{U,W}(E) = \sup \left\| \sum_{i=1}^{I} \lambda_i \mathbf{m}(E \cap E_i) \right\|_{U,W}, \quad E \in \Sigma,$$

where $\|L\|_{U,W} = \sup_{\mathbf{x} \in U} p_W(L(\mathbf{x}))$ and the supremum is taken over all finite sets of scalars $\{\lambda_i \in \mathbb{K};\ |\lambda_i| \leq 1, i = 1,2,\ldots,I\}$ and all disjoint sets $\{E_i \in \Delta;\ i = 1,2,\ldots,I\}$. Note that the scalar (U,W)-semivariation $\|\mathbf{m}\|_{U,W}$ is also a submeasure. Denote $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}} = \{\|\mathbf{m}\|_{U,W};\ (U,W) \in \mathcal{U} \times \mathcal{W}\}$. Analogously, for every $(W,V) \in \mathcal{W} \times \mathcal{V}$ we define a (W,V)-semivariation $\hat{\mathbf{n}}_{W,V} : \Xi \to [0,+\infty]$ and a scalar (W,V)-semivariation $\|\mathbf{n}\|_{W,V} : \Xi \to [0,+\infty]$ of a charge $\mathbf{n} : \nabla \to L(\mathbf{Y},\mathbf{Z})$.

Let $\mathbf{X}'$, $\mathbf{Y}'$ be the topological duals of $\mathbf{X}$,$\mathbf{Y}$, respectively. For every $y' \in \mathbf{Y}'$, $U \in \mathcal{U}$ and $E \in \Sigma$ we define the *U-variation* of the charge $y'\mathbf{m} : \Delta \to \mathbf{X}'$ by the formula

$$\mathbf{var}_U(y'\mathbf{m}, E) = \sup \sum_{i=1}^{I} |(y'\mathbf{m})(E \cap E_i)\mathbf{x}_i|,$$

where the supremum is taken over all finite pairwise disjoint sets $E_i \in \Delta$ and over all finite sets of elements $\mathbf{x}_i \in U$, $i = 1,2,\ldots,I$. Note that the (U,W)-semivariation of $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ may be expressed in the form

$$\hat{\mathbf{m}}_{U,W}(E) = \sup_{y' \in W^0} \mathbf{var}_U(y'\mathbf{m}, E), \quad E \in \Sigma,$$

where $W^0 \in \mathbf{Y}'$ denotes the polar of the set $W \in \mathcal{W}$, cf. [14].

Definition 1.2. Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. Denote by

(a) $\Delta_{U,W}$ the greatest δ-subring of Δ of subsets of finite (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$ and $\Delta_{\mathcal{U},\mathcal{W}} = \{\Delta_{U,W}; (U,W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively; the restriction of $\mathbf{m}$ to $\Delta_{U,W}$ will be denoted by $\mathbf{m}_{U,W}$;

(b) $\Delta^u_{U,W}$ the greatest δ-subring of Δ on which the restriction $\mathbf{m}_{U,W} : \Delta^u_{U,W} \to L(\mathbf{X}_U, \mathbf{Y}_W)$ of the measure $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ is uniformly countable additive with $\mathbf{m}_{U,W}(E) = \mathbf{m}(E)$ for $E \in \Delta^u_{U,W}$ and $\Delta^u_{\mathcal{U},\mathcal{W}} = \{\Delta^u_{U,W}; (U,W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively;

(c) $\Delta^c_{U,W}$ the greatest δ-subring of Δ where $\hat{\mathbf{m}}_{U,W}$ is continuous and $\Delta^c_{\mathcal{U},\mathcal{W}} = \{\Delta^c_{U,W}; (U,W) \in \mathcal{U} \times \mathcal{W}\}$ the lattice with the order given with inclusions of $U \in \mathcal{U}$ and $W \in \mathcal{W}$, respectively.

Analogously for $(W,V) \in \mathcal{W} \times \mathcal{V}$ we define $\nabla_{W,V}$, $\nabla^u_{W,V}$, $\nabla^c_{W,V}$, and $\nabla_{\mathcal{W},\mathcal{V}}$, $\nabla^u_{\mathcal{W},\mathcal{V}}$, $\nabla^c_{\mathcal{W},\mathcal{V}}$. Obviously we have

Lemma 1.3. *The lattices $\Delta^c_{\mathcal{U},\mathcal{W}}$, $\Delta^u_{\mathcal{U},\mathcal{W}}$ (resp. $\nabla^c_{\mathcal{W},\mathcal{V}}$, $\nabla^u_{\mathcal{W},\mathcal{V}}$) are sublattices of $\Delta_{\mathcal{U},\mathcal{W}}$ (resp. $\nabla_{\mathcal{W},\mathcal{V}}$).*

Denote by $\Delta_{U,W} \otimes \nabla_{W,V}$ the smallest δ-ring containing all rectangles $A \times B$, $A \in \Delta_{U,W}, B \in \nabla_{W,V}$, where $(U,W) \in \mathcal{U} \times \mathcal{W}$, $(W,V) \in \mathcal{W} \times \mathcal{V}$. If $\mathcal{D}_1$, $\mathcal{D}_2$ are two δ-rings of subsets of S, T, respectively, then clearly $\sigma(\mathcal{D}_1 \otimes \mathcal{D}_2) = \sigma(\mathcal{D}_1) \otimes \sigma(\mathcal{D}_2)$. For $E \subset S \times T$, $t \in T$, put

$$E^t = \{s \in S;\ (s,t) \in E\}.$$

For a more detailed description of the basic $L(\mathbf{X}, \mathbf{Y})$-measure set structures when both $\mathbf{X}$ and $\mathbf{Y}$ are C. B. L. C. S., cf. [14].

1.4. Basic convergences of functions

In the theory of integration in Banach spaces we suppose the generalizations of the classical notions, such as almost everywhere convergence, almost uniform convergence, and convergence in measure or semivariation of measurable functions and relations among them as commonly well-known, cf. [4]. All this theory may be generalized to C. B. L. C. S. as follows.

Let $\beta_{\mathcal{U},\mathcal{W}}$ be a lattice of submeasures $\beta_{U,W} : \Sigma \to [0, +\infty], (U,W) \in \mathcal{U} \times \mathcal{W}$, where

$$\begin{aligned} \beta_{U_2,W_2} \wedge \beta_{U_3,W_3} &= \beta_{U_2 \wedge U_3, W_2 \vee W_3}, \\ \beta_{U_2,W_2} \vee \beta_{U_3,W_3} &= \beta_{U_2 \vee U_3, W_2 \wedge W_3}, \end{aligned}$$

for $(U_2, W_2), (U_3, W_3) \in \mathcal{U} \times \mathcal{W}$, e.g. $\beta_{\mathcal{U},\mathcal{W}} = \hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$, or $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}}$.

For $(U,W) \in \mathcal{U} \times \mathcal{W}$ denote $\mathcal{O}(\beta_{U,W}) = \{N \in \Sigma;\ \beta_{U,W}(N) = 0\}$. The set $N \in \Sigma$ is called $\beta_{\mathcal{U},\mathcal{W}}$*-null* if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $N \in \mathcal{O}(\beta_{U,W})$. We say that an assertion holds $\beta_{\mathcal{U},\mathcal{W}}$*-almost everywhere*, shortly $\beta_{\mathcal{U},\mathcal{W}}$*-a.e.*, if it holds everywhere except in a $\beta_{\mathcal{U},\mathcal{W}}$-null set. A set $E \in \Sigma$ is said to be of *finite submeasure* $\beta_{\mathcal{U},\mathcal{W}}$ if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\beta_{U,W}(E) < +\infty$.

Definition 1.4. Let $E \in \Sigma$ and $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $(\mathbf{f}_n : S \to \mathbf{X})_{n=1}^{\infty}$ *of functions* (R,E)*-converges* $\beta_{U,W}$-a.e. *to a function* $\mathbf{f} : S \to \mathbf{X}$ if $\lim_{n\to\infty} p_R(\mathbf{f}_n(s) - \mathbf{f}(s)) = 0$ for every $s \in E \setminus N$, where $N \in \mathcal{O}(\beta_{U,W})$.

We say that *a sequence* $(\mathbf{f}_n : S \to \mathbf{X})_{n=1}^{\infty}$ *of functions* $(\mathcal{U}, E)$*-converges* $\beta_{\mathcal{U},\mathcal{W}}$-a.e. *to a function* $\mathbf{f} : S \to \mathbf{X}$ if there exist $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $(\mathbf{f}_n)_{n=1}^{\infty}$ of functions (R,E)-converges $\beta_{U,W}$-a.e. to $\mathbf{f}$. We write $\mathbf{f} = \mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n$ $\beta_{\mathcal{U},\mathcal{W}}$-a.e.

Definition 1.5. Let $E \in \Sigma$ and $R \in \mathcal{U}$, $(U,W) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $(\mathbf{f}_n : S \to \mathbf{X})_{n=1}^{\infty}$ *of functions* (R,E)*-converges uniformly to a function* $\mathbf{f} : S \to \mathbf{X}$, if $\lim_{n\to\infty} \|\mathbf{f}_n - \mathbf{f}\|_{E,R} = 0$, where $\|\mathbf{f}\|_{E,R} = \sup_{s \in E} p_R(\mathbf{f}(s))$.

We say that *a sequence* $(\mathbf{f}_n : S \to \mathbf{X})_{n=1}^{\infty}$ *of functions* (R,E)*-converges* $\beta_{U,W}$*-almost uniformly to a function* $\mathbf{f} : S \to \mathbf{X}$ if for every $\varepsilon > 0$ there exists a set $N \in \Sigma$, such that $\beta_{U,W}(N) < \varepsilon$ and the sequence $(\mathbf{f}_n)_{n=1}^{\infty}$ of functions $(R, E \setminus N)$-converges uniformly to $\mathbf{f}$.

We say that *a sequence* $(\mathbf{f}_n : S \to \mathbf{X})_{n=1}^{\infty}$ *of functions* $(\mathcal{U}, E)$*-converges* $\beta_{\mathcal{U},\mathcal{W}}$*-almost uniformly to a function* $\mathbf{f} : S \to \mathbf{X}$, if there exist $R \in \mathcal{U}$,

$(U,W) \in \mathcal{U}\times\mathcal{W}$, such that the sequence $(\mathbf{f}_n)_{n=1}^{\infty}$ of functions (R,E)-converges $\beta_{U,W}$-almost uniformly to $\mathbf{f}$.

For a more detail explanation of described convergences of functions in C. B. L. C. S. and relations among them, cf. [16].

1.5. Measure structures

For $(U,W) \in \mathcal{U}\times\mathcal{W}$ we say that a charge $\mathbf{m}$ is of *σ-finite (U,W)-semivariation* if there exist sets $E_n \in \Delta_{U,W}$, $n \in \mathbb{N}$, such that $S = \bigcup_{n=1}^{\infty} E_n$. For $\varphi \in \Phi$, we say that a charge $\mathbf{m}$ is of *σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if for every $U \in \mathcal{U}$ the charge $\mathbf{m}$ is of *σ-finite $(U,\varphi(U))$-semivariation.*

Definition 1.6. We say that a charge $\mathbf{m}$ is *of σ_Φ-finite $(\mathcal{U},\mathcal{W})$-semivariation* if there exists a function $\varphi \in \Phi$, such that $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation.

Let $W \in \mathcal{W}$. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *(W,σ)-additive vector measure*, if μ is a $\mathbf{Y}_W$-valued (countable additive) vector measure.

Definition 1.7. We say that a charge $\mu : \Sigma \to \mathbf{Y}$ is a *$(\mathcal{W},\sigma)$-additive vector measure*, if there exists $W \in \mathcal{W}$, such that μ is a (W,σ)-additive vector measure.

Let $W \in \mathcal{W}$ and $(\nu_n : \Sigma \to \mathbf{Y})_{n=1}^{\infty}$ be a sequence of (W,σ)-additive vector measures. If for every $\varepsilon > 0$, $E \in \Sigma$ with $p_W(\nu_n(E)) < +\infty$ for each $n \in \mathbb{N}$, and $E_i \in \Sigma$, $E_i \cap E_j = \emptyset$, $i \neq j$, $i,j \in \mathbb{N}$, there exists $J_0 \in \mathbb{N}$, such that for every $J \geq J_0$,

$$p_W\left(\nu_n\left(\bigcup_{i=J+1}^{\infty} E_i \cap E\right)\right) < \varepsilon$$

uniformly for every $n \in \mathbb{N}$, then we say that the sequence of measures $(\nu_n)_{n=1}^{\infty}$ is *uniformly (W,σ)-additive on* Σ, cf. [17].

Definition 1.8. We say that the family of measures $\nu_n : \Sigma \to \mathbf{Y}$, $n \in \mathbb{N}$, is *uniformly $(\mathcal{W},\sigma)$-additive on* Σ, if there exists $W \in \mathcal{W}$, such that the family of measures ν_n, $n \in \mathbb{N}$, is uniformly (W,σ)-additive on Σ.

The following definition is a generalization of the notion of the σ-additivity of an operator-valued measure in the strong operator topology in Banach spaces, cf. [4], to C. B. L. C. S.

Definition 1.9. Let $\varphi \in \Phi$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ is a *σ_φ-additive measure* if $\mathbf{m}$ is of σ_φ-finite $(\mathcal{U},\mathcal{W})$-semivariation, and for every $A \in \Delta_{U,\varphi(U)}$ the charge $\mathbf{m}(A \cap \cdot)\mathbf{x} : \Sigma \to \mathbf{Y}$ is a $(\varphi(U),\sigma)$-additive measure for every $\mathbf{x} \in \mathbf{X}_U$, $U \in \mathcal{U}$. We say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ is a *σ_Φ-additive measure* if there exists $\varphi \in \Phi$, such that $\mathbf{m}$ is a σ_φ-additive measure.

In what follows, $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ and $\mathbf{n} : \nabla \to L(\mathbf{Y},\mathbf{Z})$ are supposed to be operator-valued σ_Φ- and σ_Ψ-additive measures, respectively.

Definition 1.10. Let $(U,W) \in \mathcal{U} \times \mathcal{W}$. We say that a (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$ *is continuous on* Σ if for each sequence $(E_n)_{n=1}^{\infty}$ of sets from Σ, such that $E_n \searrow \emptyset$ (i.e., $E_n \supset E_{n+1}$, $\bigcap_{n=1}^{\infty} E_n = \emptyset$) with $\hat{\mathbf{m}}_{U,W}(E_1) < +\infty$ holds $\hat{\mathbf{m}}_{U,W}(E_n) \to 0$ as $n \to \infty$.

If for every couple $(U,W) \in \mathcal{U} \times \mathcal{W}$ a (U,W)-semivariation $\hat{\mathbf{m}}_{U,W}$ is continuous on Σ, then we say that a charge $\mathbf{m} : \Delta \to L(\mathbf{X},\mathbf{Y})$ is of *continuous* $(\mathcal{U},\mathcal{W})$*-semivariation*.

1.6. An integral in C. B. L. C. S.

We use $\mathcal{M}_{\Delta,\mathcal{U}}$ to denote *the space of all* $(\Delta,\mathcal{U})$*-measurable functions*, i.e. the largest vector space of functions $\mathbf{f} : S \to \mathbf{X}$ with the property: there exists $R \in \mathcal{U}$, such that for every $U \in \mathcal{U}$, $U \supset R$, and $\delta > 0$ holds $\{s \in S;\, p_U(\mathbf{f}(s)) \geq \delta\} \in \Sigma$. In what follows we deal only with $(\Delta,\mathcal{U})$-measurable functions.

Definition 1.11. A function $\mathbf{f} : S \to \mathbf{X}$ is called Δ-*simple* if $\mathbf{f}(S)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x}) \in \Delta$ for every $\mathbf{x} \in \mathbf{X} \setminus \{0\}$. Let $\mathcal{S}$ denote the space of all Δ-simple functions.

For $(U,W) \in \mathcal{U} \times \mathcal{W}$, a function $\mathbf{f} : S \to \mathbf{X}$ is said to be $\Delta_{U,W}$-*simple* if $\mathbf{f} = \sum_{i=1}^{I} \mathbf{x}_i \chi_{E_i}$, where $\mathbf{x}_i \in \mathbf{X}_U$, $E_i \in \Delta_{U,W}$, such that $E_i \cap E_j = \emptyset$, for $i \neq j$, $i,j = 1,2,\ldots,I$. The space of all $\Delta_{U,W}$-simple functions is denoted by $\mathcal{S}_{U,W}$.

A function $\mathbf{f} \in \mathcal{S}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$-*simple* if there exists a couple $(U,W) \in \mathcal{U} \times \mathcal{W}$, such that $\mathbf{f} \in \mathcal{S}_{U,W}$. The space of all $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions is denoted by $\mathcal{S}_{\mathcal{U},\mathcal{W}}$.

For every $E \in \Sigma$ and $(U,W) \in \mathcal{U} \times \mathcal{W}$, we define the integral of a $\Delta_{U,W}$-simple function $\mathbf{f} : S \to \mathbf{X}$ by the formula

$$\int_E \mathbf{f}\, \mathrm{d}\mathbf{m} = \sum_{i=1}^{I} \mathbf{m}(E \cap E_i)\mathbf{x}_i.$$

Note that for the function $\mathbf{f} \in \mathcal{S}_{U,W}$ the integral $\int_{\cdot} \mathbf{f}\, \mathrm{d}\mathbf{m}$ is a (W,σ)-additive measure on Σ.

It may be proved that $\mathcal{M}_{\Delta,\mathcal{U}} \supset \mathcal{F}_{\Delta}$, where $\mathcal{F}_{\Delta}$ is the set of functions $\mathbf{f} : S \to \mathbf{X}$, such that for $(U,W) \in \mathcal{U} \times \mathcal{W}$ there exists a sequence $(\mathbf{f}_n)_{n=1}^{\infty}$ of $\Delta_{U,W}$-simple functions U-converging on the whole S to $\mathbf{f}$. Elements of $\mathcal{F}_{\Delta}$ are called $\Delta_{U,W}$*-measurable functions* (or measurable in the sense of Dobrakov, cf. [4]).

Theorem 1.12. [cf. [17], Theorem 3.8] *Let* $\mathbf{m}$ *be a* σ_{Φ}*-additive measure and* $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$. *If there exists a sequence* $(\mathbf{f}_n)_{n=1}^{\infty}$ *of* $\Delta_{\mathcal{U},\mathcal{W}}$*-simple functions, such that*

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e.,*

(b) *the integrals* $\int_{\cdot} \mathbf{f}_n\, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, *are uniformly* $(\mathcal{W},\sigma)$*-additive measures on* Σ,

then the limit $\nu(E,\mathbf{f}) = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n\, \mathrm{d}\mathbf{m}$ *exists uniformly in* $E \in \Sigma$.

Definition 1.13. A function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ is said to be $\Delta_{\mathcal{U},\mathcal{W}}$*-integrable* if there exists a sequence $(\mathbf{f}_n)_{n=1}^{\infty}$ of $\Delta_{\mathcal{U},\mathcal{W}}$-simple functions, such that

(a) $\mathcal{U}\text{-}\lim_{n\to\infty} \mathbf{f}_n = \mathbf{f}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e.,

(b) $\int_{\cdot} \mathbf{f}_n \, \mathrm{d}\mathbf{m}$, $n \in \mathbb{N}$, are uniformly $(\mathcal{W}, \sigma)$-additive measures on Σ.

Let $\mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ denote *the family of all* $\Delta_{\mathcal{U},\mathcal{W}}$*-integrable functions*. Then *the integral* of a function $\mathbf{f} \in \mathcal{I}_{\mathcal{U},\mathcal{W},\Delta}$ on a set $E \in \Sigma$ is defined by the equality

$$\mathbf{y}_E = \int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}.$$

A criterium of integrability of a $(\Delta, \mathcal{U})$-measurable function is given in the following theorem.

Theorem 1.14. [cf. [17], Theorem 4.3] *A* $(\Delta, \mathcal{U})$*-measurable function* $\mathbf{f}$ *is* $\Delta_{\mathcal{U},\mathcal{W}}$*-integrable if and only if there exists a sequence* $(\mathbf{f}_n)_{n=1}^{\infty}$ *of* $\Delta_{\mathcal{U},\mathcal{W}}$*-simple functions, such that*

(a) $(\mathcal{U}, E)$*-converges* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e. to* $\mathbf{f}$*, and*

(b) *the limit* $\mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m} = \nu(E)$ *exists for every* $E \in \Sigma$.

In this case $\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \mathcal{W}\text{-}\lim_{n\to\infty} \int_E \mathbf{f}_n \, \mathrm{d}\mathbf{m}$ *for every set* $E \in \Sigma$ *and this limit is uniform on* Σ.

More on integrable functions and further results related to the generalized Dobrakov integral in C. B. L. C. S., see [18] and [19].

2. Bornological product measures

Bornological product of a σ_Φ-additive measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and σ_Ψ-additive measure $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ was defined in [20]. Now we recall its definition.

Definition 2.1. We say that a *bornological product measure* of a σ_Φ-additive measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and σ_Ψ-additive measure $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ exists on $\Delta \otimes \nabla$ (we write $\mathbf{m} \otimes \mathbf{n} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$), if there exists one and only one σ_Γ-additive measure $\mathbf{m} \otimes \mathbf{n} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$ defined by the formula

$$(\mathbf{m} \otimes \mathbf{n})(A \times B) = \mathbf{n}(B)\mathbf{m}(A)$$

for each $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, where there exists $\gamma \in \Gamma$, $\varphi \in \Phi$, $\psi \in \Psi$, such that $\gamma = \psi \circ \varphi$ and $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$.

Remark 2.2. Definition 2.1 differs from that of Dobrakov [6], Definition 1, in reduction to Banach spaces. Instead of the general $\Delta \otimes \nabla$ we deal only with $\Delta_{U,W} \otimes \nabla_{W,V}$, $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$. In fact, only our case is needed for proving the general Fubini theorem in [21].

The Hahn-Banach theorem and the uniqueness of enlarging of the finite scalar measure from the ring to the generated σ-ring imply that if $\mathbf{l}_1, \mathbf{l}_2 : \Delta_{U,W} \otimes \nabla_{W,V} \to L(\mathbf{X}_U, \mathbf{Z}_V)$ are two σ_γ-additive measures ($\gamma \in \Gamma$), such that $\mathbf{l}_1(A \times B) = \mathbf{l}_2(A \times B)$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, then $\mathbf{l}_1 = \mathbf{l}_2$ on $\Delta_{U,W} \otimes \nabla_{W,V}$.

Remark 2.3. The bornological product measure is a complicated object from the reason of the following implications: if $(U_1, W_1, V_1), (U_2, W_2, V_2) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$, then

$$(U_1, W_1) \ll (U_2, W_2) \Rightarrow \Delta_{U_2,W_2} \subset \Delta_{U_1,W_1},$$
$$(W_1, V_1) \ll (W_2, V_2) \Rightarrow \nabla_{W_2,V_2} \subset \nabla_{W_1,V_1}.$$

In general, for a fixed $W \in \mathcal{W}$,

$$(U_1, V_1) \ll (U_2, V_2) \Rightarrow \Delta_{U_2,W} \otimes \nabla_{W,V_2} \subset \Delta_{U_1,W} \otimes \nabla_{W,V_1}$$

and we may say nothing about the uniqueness, the existence, etc. of $W \in \mathcal{W}$.

The following lemma gives sufficient conditions for the existence of bornological product measure. Further results related to bornological product measures may be found in our recent paper [20] and those related to general Fubini theorem in [21].

Lemma 2.4. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$, such that $V \subseteq \psi(W)$, $W \subseteq \varphi(U)$, $\gamma(U) \subset \psi(\varphi(U))$, where $\gamma \in \Gamma$, $\varphi \in \Phi$, $\psi \in \Psi$ and $\gamma = \psi \circ \varphi$. If for every $\mathbf{x} \in \mathbf{X}_U$ there exists a $\mathbf{Z}_V$-valued vector measure $\mathbf{l}_{\mathbf{x}}$ on $\Delta_{U,W} \otimes \nabla_{W,V}$, such that*

$$\mathbf{l}_{\mathbf{x}}(A \times B) = \mathbf{n}_{W,V}(B)\mathbf{m}_{U,W}(A)\mathbf{x}$$

for every $A \in \Delta_{U,W}$ and $B \in \nabla_{W,V}$, then the product measure $\mathbf{m} \otimes \mathbf{n}$ exists on $\Delta \otimes \nabla$.

Proof. For $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$ put

$$(\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V})(E)\mathbf{x} = \mathbf{l}_{\mathbf{x}}(E).$$

Indeed, we have to prove that

(a) $\mathbf{l}_{\alpha\,\mathbf{x}_1+\beta\,\mathbf{x}_2}(E) = \alpha\,\mathbf{l}_{\mathbf{x}_1}(E) + \beta\,\mathbf{l}_{\mathbf{x}_2}(E)$, and
(b) $\lim_{\mathbf{x}\to 0} \mathbf{l}_{\mathbf{x}}(E) = 0$,

for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$, $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}_U$ and all scalars $\alpha, \beta \in \mathbb{K}$.

Denote by $\mathcal{R}$ the ring of all finite unions of rectangles of the form $A \times B$, where $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$. Denote by

$$\mathbf{var}_V(z'\mathbf{l}_{\mathbf{x}}, \cdot) : \Delta_{U,W} \otimes \nabla_{W,V} \to [0, +\infty]$$

the V-variation of the real measure $z'\mathbf{l}_{\mathbf{x}} : \Delta_{U,W} \otimes \nabla_{W,V} \to [0, +\infty]$, where $z' \in V^0$. We will use the following fact:

(c) Let $z' \in V^0$ and $E \in \Delta_{U,W} \otimes \nabla_{W,V}$. Then the inequality

$$|\langle \mathbf{l}_{\mathbf{x}}(E_1) - \mathbf{l}_{\mathbf{x}}(E_2), z' \rangle| \le \mathbf{var}_V(z'\mathbf{l}_{\mathbf{x}}, E_1 \triangle E_2),$$

for $E_1, E_2 \in \Delta_{U,W} \otimes \nabla_{W,V}$, and [13], Theorem D, § 13, imply that for every $\varepsilon > 0$ there exists a set $F \in \mathcal{R}$, such that

$$|\langle \mathbf{l}_{\mathbf{x}}(E) - \mathbf{l}_{\mathbf{x}}(F), z' \rangle| < \varepsilon.$$

Let $\alpha, \beta \in \mathbb{K}$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{X}_U$ be given.

Since $\mathbf{l}_{\mathbf{x}}(A \times B) = \mathbf{n}_{W,V}(B)\mathbf{m}_{U,W}(A)\mathbf{x}$ for every $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, the values $\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V}$ are linear operators and $\mathbf{l}_{\mathbf{x}}$ is an additive function, then (a) holds for $E \in \mathcal{R}$. From (c) and the Hahn-Banach theorem for Banach spaces it follows that (a) holds for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$.

To show that (b) holds, let $E \in \Delta_{U,W} \otimes \nabla_{W,V}$ and consider $A \in \Delta_{U,W}$, $B \in \nabla_{W,V}$, such that $E \subset A \times B$. Let $F \in \mathcal{R} \cap (A \times B)$. Without loss of generality we may suppose that $F = \bigcup_{i=1}^{I}(A_i \times B_i)$, where $A_i \in \Delta_{U,W}$, $B_i \in \nabla_{W,V}$ and the B_i's are pairwise disjoint sets, $i = 1, 2, \dots, I$. Then

$$|\langle \mathbf{l}_{\mathbf{x}}(F), z' \rangle| \leq p_V(\mathbf{l}_{\mathbf{x}}(F)) = p_V\left(\sum_{i=1}^{I} \mathbf{l}_{\mathbf{x}}(A_i \times B_i)\right) = p_V\left(\sum_{i=1}^{I} \mathbf{n}(B_i)\mathbf{m}(A_i)\mathbf{x}\right)$$
$$\leq p_U(\mathbf{x}) \cdot \|\mathbf{m}\|_{U,W}(A) \cdot \hat{\mathbf{n}}_{W,V}(B)$$

for every $z' \in V^0$. Since $B \in \nabla_{W,V}$, then $\hat{\mathbf{n}}_{W,V}(B) < +\infty$, and the uniform boundedness principle implies that

$$\|\mathbf{m}\|_{U,W}(A) = \sup_{\mathbf{x} \in U} \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}(A) = \sup_{\mathbf{x} \in U} \sup_{y' \in W^0} \mathbf{var}_W(y'\mathbf{m}(\cdot)\mathbf{x}, A) < +\infty.$$

Thus,

$$\lim_{\mathbf{x} \to \mathbf{0}} |\langle \mathbf{l}_{\mathbf{x}}(F), z' \rangle| = 0$$

uniformly for $F \in \mathcal{R} \cap (A \times B)$ and $z' \in V^0$, $V \in \mathcal{V}$. Using (c) we easily obtain (b) for every $E \in \Delta_{U,W} \otimes \nabla_{W,V}$. □

Lemma 2.5. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Then*

(i) for every $E \in \Delta_{U,W} \otimes \nabla^c_{W,V}$ and every $\mathbf{x} \in \mathbf{X}_U$ the function $t \mapsto \mathbf{m}(E^t)\mathbf{x}$, $t \in T$, is $\mathfrak{B}_{\mathbf{Y}}$-bounded and $\nabla^c_{W,V}$-measurable;

(ii) for every $E \in \Delta^u_{U,W} \otimes \nabla^c_{W,V}$ the function $t \mapsto \|\mathbf{m}(E^t)\|_{U,W}$, $t \in T$, is bounded and $\nabla^c_{W,V}$-measurable;

(iii) for every $E \in \Delta^c_{U,W} \otimes \nabla^c_{W,V}$ the function $t \mapsto \hat{\mathbf{m}}_{U,W}(E^t)$, $t \in T$, is bounded and $\nabla^c_{W,V}$-measurable.

Proof. Let us prove the item (i). Suppose that $E \in \Delta_{U,W} \otimes \nabla^c_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$. Take $A \in \Delta_{U,W}$ and $B \in \nabla^c_{W,V}$, such that $E \subset A \times B$. Denote by $\mathcal{M}$ the class of all sets $N \in \Delta_{U,W} \otimes \nabla_{W,V} \cap (A \times B)$ for which (i) holds. Then clearly $\mathcal{M}$ contains the ring $\mathcal{R} \cap (A \times B)$, where $\mathcal{R}$ is the ring of all finite unions of pairwise disjoint rectangles $A_1 \times B_1$, where $A_1 \in \Delta_{U,W}, B_1 \in \nabla_{W,V}$. Since

$$\sup_{t \in T} p_W(\mathbf{m}(N^t)\mathbf{x}) \leq \|\mathbf{m}(\cdot)\mathbf{x}\|_{U,W}(A) < +\infty$$

for every $N \in \mathcal{M}$ and since each $\nabla^c_{W,V}$-measurable function belongs to the closure of the pointwise limits in the topology of $\mathbf{X}_U$, $U \in \mathcal{U}$, then the σ-additivity of the measure $\mathbf{m}(\cdot)\mathbf{x}$ on $\Delta_{U,W}$ implies that $\mathcal{M}$ is a monotone class of sets. By [13], Theorem B, § 6, we have that $\mathcal{M} = \Delta_{U,W} \otimes \nabla^c_{W,V} \cap (A \times B)$, and therefore $E \in \mathcal{M}$.

The assertions (ii) and (iii) may be proved analogously using the continuity and finiteness of semivariations $\|\mathbf{m}\|_{U,W}$ on $\Delta^u_{U,W}$ and $\hat{\mathbf{m}}_{U,W}$ on $\Delta^c_{U,W}$, respectively. □

Remark 2.6. From the proof it is clear that the result of Lemma 2.5 may be formulated more generally replacing $\nabla^c_{W,V}$ by an arbitrary δ-ring $\mathcal{D}$ of subsets of T (when considering $\mathcal{D}$-measurability of functions in the sense of Dobrakov, i.e. $\mathcal{D}$-measurable functions are closed under the formation of pointwise limits of sequences).

Let $(W, V) \in \mathcal{W} \times \mathcal{V}$. For a $\nabla_{W,V}$-measurable function $\mathbf{g} : T \to \mathbf{Y}_W$ we define the submeasure $\hat{\mathbf{n}}_{W,V}(\mathbf{g}, B)$ for $B \in \sigma(\nabla_{W,V})$ as follows:

$$\hat{\mathbf{n}}_{W,V}(\mathbf{g}, B) = \sup\left\{p_V\left(\int_B \mathbf{h}\,\mathrm{d}\mathbf{n}\right)\right\},$$

where the supremum is taken over all $\mathbf{h} \in \mathcal{S}_{W,V}$, and $p_W(\mathbf{h}(t)) \le p_W(\mathbf{g}(t))$ for each $t \in T$. Let us denote by $L^1_{W,V}(\mathbf{n})$ the space of all $\nabla_{W,V}$-integrable functions with the bounded and continuous seminorm $\hat{\mathbf{n}}_{W,V}(\cdot, B)$. Analogously for $(U, W) \in \mathcal{U} \times \mathcal{W}$ we define $\hat{\mathbf{m}}_{U,W}(\cdot, A)$ and the space $L^1_{U,W}(\mathbf{m})$. For more information on $L^1_{U,W}$-gauge and its properties, see [19].

Using the above stated lemmas we may prove the following properties of product measures.

Theorem 2.7. *Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$. Then*

(i) *the product measure $\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V}$ exists on $\Delta_{U,W} \otimes \nabla^c_{W,V}$;*
(ii) *$\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V}$ is a σ-additive vector measure in the uniform topology of the space $L(\mathbf{X}_U, \mathbf{Z}_V)$ on $\Delta^u_{U,W} \otimes \nabla^c_{W,V}$;*
(iii) *the semivariation $(\widehat{\mathbf{m} \otimes \mathbf{n}})_{U,V}$ is continuous on $\Delta^c_{U,W} \otimes \nabla^c_{W,V}$.*

Proof. (i) Let $E \in \Delta_{U,W} \otimes \nabla^c_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$. Lemma 2.5(i) implies that the function $t \mapsto \mathbf{m}(E^t)\mathbf{x}$, $t \in T$, is $\mathfrak{B}_{\mathbf{Y}}$-bounded and $\nabla^c_{W,V}$-measurable. Since

$$\{t \in T;\ \mathbf{m}(E^t)\mathbf{x} \neq 0\} \in \nabla^c_{W,V},$$

and since the (W, V)-semivariation $\hat{\mathbf{n}}_{W,V}$ is continuous on $\nabla^c_{W,V}$, then by Theorem 3.6 in [18], the function $t \mapsto \mathbf{m}_{U,W}(E^t)\mathbf{x}$, $t \in T$, is $\nabla^c_{W,V}$-integrable. Since $E \in \Delta_{U,W} \otimes \nabla^c_{W,V}$ and $\mathbf{x} \in \mathbf{X}_U$ are arbitrary, by Theorem 2.4 in [20] the product measure $\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V}$ exists on $\Delta_{U,W} \otimes \nabla^c_{W,V}$.

(ii) It is easy to see that the product measure $\mathbf{m}_{U,W} \otimes \mathbf{n}_{W,V}$ is σ-additive in the uniform topology of the space $L(\mathbf{X}_U, \mathbf{Z}_V)$ on $\Delta^u_{U,W} \otimes \nabla^c_{W,V}$ if and only if $(E_n)_{n=1}^\infty \in \Delta^u_{U,W} \otimes \nabla^c_{W,V}$ with $E_n \searrow \emptyset$ implies that $\|\mathbf{m} \otimes \mathbf{n}\|_{U,V}(E_n) \to 0$ as $n \to \infty$.

Let $(E_n)_{n=1}^\infty$ be a sequence of sets from $\Delta^u_{U,W} \otimes \nabla^c_{W,V}$ such that $E_n \searrow \emptyset$. By Lemma 2.5(ii) the functions $t \mapsto \|\mathbf{m}\|_{U,W}(E^t_n)$, $t \in T$, $n \in \mathbb{N}$, are bounded and $\nabla^c_{W,V}$-measurable. Since $\{t \in T;\ \|\mathbf{m}\|_{U,W}(E^t_1) \neq 0\} \in \nabla^c_{W,V}$, the involved functions belong to the class $L^1_{W,V}(\mathbf{n})$.

Since $\mathbf{m}_{U,W} : \Delta^u_{U,W} \to L(\mathbf{X}_U, \mathbf{Y}_W)$ is uniformly σ-additive, and since $E^t_n \in \Delta^u_{U,W}$ for every $t \in T$ and $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \|\mathbf{m}\|_{U,W}(E^t_n) = 0$$

for every $t \in T$. Then by Theorem 17 in [5] (Lebesgue dominated convergence theorem) and Theorem 2.6 in [20] we get

$$\|\mathbf{m} \otimes \mathbf{n}\|_{U,V}(E_n) \leq \hat{\mathbf{n}}_{W,V}(\|\mathbf{m}\|_{U,W}(E_n^t), T) \to 0 \quad \text{as } n \to \infty.$$

The proof of assertion (iii) is analogous to the second one. □

3. Domination and bornological product measures

Definition 3.1. Let $(U, W) \in \mathcal{U} \times \mathcal{W}$. A non-negative finite measure $\mu_{U,W} : \Sigma \to [0, +\infty)$ is called a *bornological control measure* for (U, W)-semivariation $\hat{\mathbf{m}}_{U,W}$ iff

$$\lim_{\mu_{U,W}(E)\to 0} \hat{\mathbf{m}}_{U,W}(E) = 0, \quad E \in \Sigma.$$

We write $\hat{\mathbf{m}}_{U,W} \prec \mu_{U,W}$. If for each $(U, W) \in \mathcal{U} \times \mathcal{W}$ there exists a bornological control measure $\mu_{U,W}$ for $\hat{\mathbf{m}}_{U,W}$, then we say that *a charge* $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ *is dominated* by the system $\{\mu_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ of bornological control measures.

Analogously, for $(W, V) \in \mathcal{W} \times \mathcal{V}$ we denote by $\nu_{W,V} : \Xi \to [0, +\infty)$ a bornological control measure for $\hat{\mathbf{n}}_{W,V} : \Xi \to [0, +\infty)$ and by $\{\nu_{W,V}; (W, V) \in \mathcal{W} \times \mathcal{V}\}$ a system of bornological control measures for an operator-valued measure $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$.

Note that the condition in Definition 3.1 is sometimes known as continuity, or absolute continuity of one measure with respect to another one, cf. [4, 8]. Moreover, in [9], Theorem 5, the following result is also proved (rewritten in our setting).

Lemma 3.2. *If* $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ *is dominated, then for each* $(U, W) \in \mathcal{U} \times \mathcal{W}$ *there exists a non-negative finite measure* $\mu_{U,W}$ *on* Σ*, such that*

$$\hat{\mathbf{m}}_{U,W}(E) \to 0 \ \text{ if and only if } \ \mu_{U,W}(E) \to 0, \quad E \in \Sigma.$$

Remark 3.3. On the strength of Lemma 3.2 we can choose bornological control measures, such that $\mu_{U,W} \prec \hat{\mathbf{m}}_{U,W}$ provided they exist. In terminology used in [8] such measures are called *equivalent*.

The notion of continuity of semivariation of measure is needed in many occasions in the integration theory with respect to the operator-valued measure, cf. [4, 5], e.g. convergence theorems are based on this notion in countable additive case of operator-valued measure countable additive in the strong operator topology. The continuity of operators $\mathbf{m}(E) \in L(\mathbf{X}, \mathbf{Y})$, $E \in \Delta$, is clearly a necessary condition for the continuity of $\hat{\mathbf{m}}_{U,W}$, $(U, W) \in \mathcal{U} \times \mathcal{W}$, but not a sufficient one. However, from the Orlicz-Pettis theorem, see Theorem 5 in [5], it follows that if $\mathbf{Y}_W$ is weakly complete (more generally, if $\mathbf{Y}_W$ contains no subspace isomorphic to the space c_0) and $\hat{\mathbf{m}}_{U,W}$ is bounded on Σ, then $\hat{\mathbf{m}}_{U,W}$ is continuous on Σ. A sufficient condition for the boundedness of $\hat{\mathbf{m}}_{U,W}$, resp. $\hat{\mathbf{n}}_{W,V}$, is as follows.

Lemma 3.4. *Let* $(U, W) \in \mathcal{U} \times \mathcal{W}$*,* $(W, V) \in \mathcal{W} \times \mathcal{V}$*.*

(a) *If* $\hat{\mathbf{m}}_{U,W} \prec \mu_{U,W}$*, then* $\hat{\mathbf{m}}_{U,W}(S) < +\infty$*.*
(b) *If* $\hat{\mathbf{n}}_{W,V} \prec \nu_{W,V}$*, then* $\hat{\mathbf{n}}_{W,V}(T) < +\infty$*.*

For the proof see [27], Lemma 5 (also [5], Corollary of Theorem 5). In connection with continuity of dominated measures I. Dobrakov proved in [7], Lemma 2, the following result (in our terminology).

Lemma 3.5. *A charge* $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ *is of continuous* $(\mathcal{U}, \mathcal{W})$*-semivariation if and only if* $\mathbf{m}$ *is dominated by the system* $\{\mu_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ *of bornological control measures, such that* $\mu_{U,W}(E) \le \|\mathbf{m}\|_{U,W}(E)$ *for each* $(U, W) \in \mathcal{U} \times \mathcal{W}$*,* $E \in \Sigma$*.*

For the sake of completeness we give short proof of Lemma 3.5 here.

Proof. The necessity is obvious. Let us suppose that $\mathbf{m}$ is of continuous $(\mathcal{U}, \mathcal{W})$-semivariation, i.e. $\hat{\mathbf{m}}_{U,W}$ is continuous on Σ for each $(U, W) \in \mathcal{U} \times \mathcal{W}$. Since $\|\mathbf{m}\|_{U,W}(E) \le \hat{\mathbf{m}}_{U,W}(E)$ for every $E \in \Sigma$, a charge $\mathbf{m}$ is countable additive in the uniform operator topology on Σ. Then by Theorem IV.10.5 in [11] for each $(U, W) \in \mathcal{U} \times \mathcal{W}$ there exists a non-negative countably additive measure $\mu_{U,W}$ on Σ, such that $\mu_{U,W}(E) \le \|\mathbf{m}\|_{U,W}(E)$, $E \in \Sigma$, and $\|\mathbf{m}\|_{U,W} \prec \mu_{U,W}$. Clearly, if $N \in \mathcal{O}(\mu_{U,W})$, then $N \in \mathcal{O}(\|\mathbf{m}\|_{U,W})$ and also $N \in \mathcal{O}(\hat{\mathbf{m}}_{U,W})$.

On the contrary, let us suppose that there exists $(U, W) \in \mathcal{U} \times \mathcal{W}$, such that

$$\lim_{\mu_{U,W}(E)\to 0} \hat{\mathbf{m}}_{U,W}(E) \neq 0, \quad E \in \Sigma.$$

Then there exists an $\varepsilon > 0$ and a sequence $(A_k)_{n=1}^{\infty}$ of sets from Σ such that $\mu_{U,W}(A_k) < 2^{-k}$ and $\hat{\mathbf{m}}_{U,W}(A_k) > \varepsilon$. Put

$$B_k = \bigcup_{i=k}^{\infty} A_k, \quad \text{and} \quad B = \bigcap_{k=1}^{\infty} B_k.$$

Since $\mu_{U,W}$ is a finite countably additive non-negative measure on Σ, then we have $\mu_{U,W}(B) = 0$, but for sufficiently large k from monotonicity and continuity of $\hat{\mathbf{m}}_{U,W}$ on Σ we have

$$\hat{\mathbf{m}}_{U,W}(B) \ge \hat{\mathbf{m}}_{U,W}(B_k) - \hat{\mathbf{m}}_{U,W}(B \setminus B_k) > \varepsilon,$$

a contradiction. Thus we have proved the existence of $\mu_{U,W}$ required. □

So, in order to guarantee the continuity of $\hat{\mathbf{m}}_{U,W}$ on Σ it is necessary and sufficient to consider such bornological control measures $\mu_{U,W}$ for which $\mu_{U,W}(E) \le \hat{\mathbf{m}}_{U,W}(E)$, $E \in \Sigma$. In what follows we consider only that case although it is not explicitly stated. Note that in Lemma 3.5 the boundedness of $\hat{\mathbf{m}}_{U,W}$ on Σ is not assumed since it follows immediately from domination, i.e. Lemma 3.4.

The bornological product measure $\mathbf{m} \otimes \mathbf{n}$ may be extended to $\delta(\Delta \otimes \nabla)$ by a standard method. In what follows denote by $\Sigma \otimes \Xi$ the σ-algebra over $\delta(\Delta \otimes \nabla)$. From examples of Banach spaces, cf. e.g. [27], $\mathbf{m} \otimes \mathbf{n}$ may fail to be countably additive on $\Delta \otimes \nabla$ even though $\mathbf{m}$ and $\mathbf{n}$ are countably additive (in the uniform and thus strong operator topology). A sufficient condition for the countable additivity of $\mathbf{m} \otimes \mathbf{n}$ gives the following theorem, cf. [27], Theorem 6.

Theorem 3.6. *Let* $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ *and* $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ *be dominated by the systems of bornological control measures* $\{\mu_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ *and* $\{\nu_{W,V}; (W, V) \in \mathcal{W} \times \mathcal{V}\}$*, respectively. Then the product* $\mathbf{m} \otimes \mathbf{n} : \Delta \otimes \nabla \to L(\mathbf{X}, \mathbf{Z})$ *has a unique extension on* $\Sigma \otimes \Xi$ *countably additive in the strong operator topology.*

In the following we obtain an explicit expression for the bornological product measure using the generalized Dobrakov integral in C. B. L. C. S. For every $G \in \Sigma \otimes \Xi$ define the function $g^G : T \to L(\mathbf{X}, \mathbf{Y})$ by the formula

$$g^G(t) = \mathbf{m}(G^t), \text{ where } G^t = \{s \in S; (s, t) \in G\}.$$

If $G = E \times F \in \Sigma \otimes \Xi$, then

$$g^{E\times F} = \mathbf{m}(E)\chi_F : T \to L(\mathbf{X}, \mathbf{Y}).$$

If $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ is dominated by the system of bornological control measures $\{\nu_{W,V}; (W, V) \in \mathcal{W} \times \mathcal{V}\}$, then by Lemma 3.4 $\hat{\mathbf{n}}_{W,V}(T) < +\infty$ for each $(W, V) \in \mathcal{W} \times \mathcal{V}$. In this case $g^{E\times F} \in \mathcal{S}_{W,V}$ and

$$\int_T g^{E\times F} \mathrm{d}\mathbf{n} = \mathbf{n}(F)\mathbf{m}(E) = (\mathbf{m} \otimes \mathbf{n})(E \times F).$$

Moreover,

$$g^{E\times F}(t) = \mathbf{m}(G^t) = \int_S \chi_{E\times F}(s, t)\, \mathrm{d}\mathbf{m}(s),$$

and

$$(\mathbf{m} \otimes \mathbf{n})(E \times F) = \int_T \left(\int_S \chi_{E\times F}(s, t)\, \mathrm{d}\mathbf{m}(s) \right) \mathrm{d}\mathbf{n}(t) = \int_T g^{E\times F} \mathrm{d}\mathbf{n}.$$

If G and H are disjoint sets in $\Sigma \otimes \Xi$, then $g^{G\cup H} = g^G + g^H$. If $(G_n)_{n=1}^\infty$ is a monotone sequence of sets in $\Sigma \otimes \Xi$ and $G = \lim_{n\to\infty} G_n$, then g^{G_n} converges to g^G as $n \to \infty$. Indeed, clearly $(G_n^t \setminus G^t) \searrow \emptyset$, and from the continuity of $\hat{\mathbf{m}}_{U,W}$ we have $\hat{\mathbf{m}}_{U,W}((G_n^t \setminus G^t)) \to 0$ as $n \to \infty$.

If $G = \bigcup_{i=1}^I E_i \times F_i$ is a disjoint representation of G where $E_i \times F_i \in \Sigma \otimes \Xi$, then

$$g^G(t) = \mathbf{m}(G^t) = \int_S \chi_G(s, t)\, \mathrm{d}\mathbf{m}(s),$$

the function g^G is $\nabla_{W,V}$-simple, and we have

$$\begin{aligned}(\mathbf{m} \otimes \mathbf{n})(G) &= \sum_{i=1}^I \mathbf{n}(F_i)\mathbf{m}(E_i) = \int_T \left(\int_S \chi_G(s, t)\, \mathrm{d}\mathbf{m}(s) \right) \mathrm{d}\mathbf{n}(t) \\ &= \int_T g^G(t)\, \mathrm{d}\mathbf{n}(t).\end{aligned}$$

Let $\mathcal{R}$ denote the class of all sets $C \in \Sigma \otimes \Xi$, such that g^C is defined on T, $g^C \in \mathcal{S}_{W,V}$, and

$$(\mathbf{m} \otimes \mathbf{n})(C) = \int_T g^C \mathrm{d}\mathbf{n}.$$

The class $\mathcal{R}$ contains semiring $\Delta \otimes \nabla$, i.e. $\Delta \otimes \nabla \subset \mathcal{R}$. If $(C_n)_{n=1}^{\infty}$ is a monotone sequence of sets from $\mathcal{R}$, then $C = \bigcup_{n=1}^{\infty} C_n \in \mathcal{R}$. Then $(g^{C_n})_{n=1}^{\infty}$ is a sequence of $\nabla_{W,V}$-integrable functions converging to g^C, thus g^C is $\nabla_{W,V}$-measurable. Since for all $t \in T$ we have $p_W(g^{C_n}(t)) \leq M < +\infty$, $n = 1, 2, \ldots$, then g^C is $\nabla_{W,V}$-integrable. According to the bounded convergence theorem we have

$$\int_T g^C(t)\,\mathrm{d}\mathbf{n}(t) = \lim_{n\to\infty} \int_T g^{C_n}(t)\,\mathrm{d}\mathbf{n}(t).$$

If $C \in \mathcal{R}$, then $(S \times T) \setminus C \in \mathcal{R}$ because $g^{(S\times T)\setminus C} = g^{S\times T} - g^C$. By lemma on monotone classes we have $\mathcal{R} = \Sigma \otimes \Xi$ and we have just proved the following theorem.

Theorem 3.7. *Let $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ be measures dominated by the systems of bornological control measures $\{\mu_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ and $\{\nu_{W,V}; (W, V) \in \mathcal{W} \times \mathcal{V}\}$, respectively. Then for each $G \in \Sigma \otimes \Xi$ the function g^G given by*

$$g^G(t) = \mathbf{m}(G^t) = \int_S \chi_G(s, t)\,\mathrm{d}\mathbf{m}(s)$$

is defined on T, $\nabla_{W,V}$-measurable, $\nabla_{W,V}$-integrable, and

$$(\mathbf{m} \otimes \mathbf{n})(G) = \int_T \mathbf{m}(G^t)\,\mathrm{d}\mathbf{n}(t),$$

i.e.

$$(\mathbf{m} \otimes \mathbf{n})(G) = \int_T \left(\int_S \chi_G(s, t)\mathrm{d}\mathbf{m}(s) \right) \mathrm{d}\mathbf{n}(t).$$

The following theorem solves the question whether the bornological product measure $\mathbf{m} \otimes \mathbf{n}$ of dominated measures $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ is also dominated.

Theorem 3.8. *Let $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$ and $\mathbf{n} : \nabla \to L(\mathbf{Y}, \mathbf{Z})$ be measures dominated by the systems of control measures $\{\mu_{U,W}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$ and $\{\nu_{W,V}; (W, V) \in \mathcal{W} \times \mathcal{V}\}$, respectively. Then there exists the (bornological) product measure $\mathbf{m} \otimes \mathbf{n} : \Sigma \otimes \Xi \to L(\mathbf{X}, \mathbf{Z})$ dominated by the system of bornological control measures $\{\mu_{U,W} \otimes \nu_{W,V}; (U, W) \in \mathcal{U} \times \mathcal{W}\}$.*

Proof. Let $(U, W, V) \in \mathcal{U} \times \mathcal{W} \times \mathcal{V}$ be fixed. Let $\alpha > 0$ and $\beta > 0$ be two real numbers, such that $\mu_{U,W}(E) < \beta$, $E \in \Sigma$, implies $\hat{\mathbf{m}}_{U,W}(E) < \alpha$, and $\nu_{W,V}(F) < \beta$, $F \in \Xi$, implies $\hat{\mathbf{n}}_{W,V}(F) < \alpha$. We will show that the condition

$$(\mu_{U,W} \otimes \nu_{W,V})(G) < \beta^2, \quad G \in \Sigma \otimes \Xi,$$

implies

$$(\widehat{\mathbf{m} \otimes \mathbf{n}})_{U,V}(G) \leq \alpha(\hat{\mathbf{m}}_{U,W}(S) + \hat{\mathbf{n}}_{W,V}(T)).$$

Put

$$A = \{t \in T; \mu_{U,W}(G^t) < \beta\}.$$

Then

$$\begin{aligned}\beta^2 &> (\mu_{U,W} \otimes \nu_{W,V})(G) = \int_T \mu_{U,W}(G^t)\,\mathrm{d}\nu_{W,V}(t) \\ &\geq \int_{T\setminus A} \mu_{U,W}(G^t)\,\mathrm{d}\nu_{W,V}(t) \geq \beta \cdot \nu_{W,V}(T \setminus A),\end{aligned}$$

from which results $\nu_{W,V}(T \setminus A) < \beta$, and therefore by assumption $\hat{\mathbf{n}}_{W,V}(T \setminus A) < \alpha$.

For an arbitrary partition $G = \bigcup_{i=1}^I G_i$, where $G_i \in \Sigma \otimes \Xi$ are disjoint sets, and for arbitrary $\mathbf{x}_i \in \mathbf{X}_U$ with $p_U(\mathbf{x}_i) \leq 1$ $(i = 1, 2, \ldots, I)$, we have by Lemma 3.4

$$p_W\left(\sum_{i=1}^I \mathbf{m}((G_i)^t)\mathbf{x}_i\right) \leq \hat{\mathbf{m}}_{U,W}(G^t) \leq \hat{\mathbf{m}}_{U,W}(S) < +\infty$$

for each $t \in T$. So by Theorem 3.7 the function

$$t \mapsto \sum_{i=1}^I \mathbf{m}((G_i)^t)\mathbf{x}_i$$

is defined on T, $\nabla_{W,V}$-measurable, $\nabla_{W,V}$-integrable and by Lemma 3.3 in [19] we have

$$\begin{aligned}p_V\left(\int_A \left[\sum_{i=1}^I \mathbf{m}((G_i)^t)\mathbf{x}_i\right] \mathrm{d}\mathbf{n}(t)\right) &\leq \sup_{t\in A} p_W\left(\sum_{i=1}^I \mathbf{m}((G_i)^t)\mathbf{x}_i\right) \cdot \hat{\mathbf{n}}_{W,V}(A) \\ &\leq \sup_{t\in A} \hat{\mathbf{m}}_{U,W}(G^t) \cdot \hat{\mathbf{n}}_{W,V}(A) \\ &\leq \alpha \cdot \hat{\mathbf{n}}_{W,V}(A),\end{aligned}$$

where for each $t \in A$ the fact that $\mu_{U,W}(G^t) < \beta$ implies $\hat{\mathbf{m}}_{U,W}(G^t) < \alpha$ is used. Further,

$$\begin{aligned}&p_V\left(\int_{T\setminus A} \left[\sum_{i=1}^I \mathbf{m}((G_i)^t)\mathbf{x}_i\right] \mathrm{d}\mathbf{n}(t)\right) \\ &\leq \sup_{t\in T\setminus A} p_W\left(\sum_{i=1}^I \mathbf{m}((G_i)^t)\mathbf{x}_i\right) \cdot \hat{\mathbf{n}}_{W,V}(T \setminus A) \\ &\leq \sup_{t\in T\setminus A} \hat{\mathbf{m}}_{U,W}(G^t) \cdot \hat{\mathbf{n}}_{W,V}(T \setminus A) \\ &\leq \hat{\mathbf{m}}_{U,W}(S)\hat{\mathbf{n}}_{W,V}(T \setminus A) \leq \alpha \cdot \hat{\mathbf{m}}_{U,W}(S).\end{aligned}$$

By Theorem 3.7 and the above estimations we get

$$p_V\left(\sum_{i=1}^{I}(\mathbf{m}\otimes\mathbf{n})(G_i)\mathbf{x}_i\right) = p_V\left(\sum_{i=1}^{I}\left[\int_T \mathbf{m}((G_i)^t)\,\mathrm{d}\mathbf{n}(t)\right]\mathbf{x}_i\right)$$

$$= p_V\left(\int_T\left[\sum_{i=1}^{I}\mathbf{m}((G_i)^t)\mathbf{x}_i\right]\mathrm{d}\mathbf{n}(t)\right)$$

$$\leq p_V\left(\int_A\left[\sum_{i=1}^{I}\mathbf{m}((G_i)^t)\mathbf{x}_i\right]\mathrm{d}\mathbf{n}(t)\right) + p_V\left(\int_{T\setminus A}\left[\sum_{i=1}^{I}\mathbf{m}((G_i)^t)\mathbf{x}_i\right]\mathrm{d}\mathbf{n}(t)\right)$$

$$\leq \alpha\cdot\hat{\mathbf{n}}_{W,V}(A) + \alpha\cdot\hat{\mathbf{m}}_{U,W}(S) \leq \alpha\cdot\hat{\mathbf{n}}_{W,V}(T) + \alpha\cdot\hat{\mathbf{m}}_{U,W}(S).$$

Since G_i are arbitrary, from it follows that

$$(\widehat{\mathbf{m}\otimes\mathbf{n}})_{U,V}(G) \leq \alpha(\hat{\mathbf{m}}_{U,W}(S) + \hat{\mathbf{n}}_{W,V}(T)).$$

This completes the proof. □

4. The Fubini-type theorem for dominated bornological product measures

In Lemma 4.1 and Theorem 4.2 we will suppose that $(\widehat{\mathbf{m}\otimes\mathbf{n}})_{U,V}$-null sets and $\mu_{U,W}\otimes\nu_{W,V}$-null sets coincide. The result of Lemma 4.1 is well-known for scalar measures. Since for dominated measure $\mathbf{m}$ (resp. $\mathbf{n}$) we may suppose by Lemma 3.2 that $\hat{\mathbf{m}}_{U,W}$-null sets and $\mu_{U,W}$-null sets (resp. $\hat{\mathbf{n}}_{W,V}$-null sets and $\nu_{W,V}$-null sets) coincide, then Lemma 4.1 holds also for dominated measures.

Lemma 4.1. *Let $(U,W,V)\in\mathcal{U}\times\mathcal{W}\times\mathcal{V}$ and $H\in\mathcal{O}((\widehat{\mathbf{m}\otimes\mathbf{n}})_{U,V})$ (or, equivalently, $H\in\mathcal{O}(\mu_{U,W}\otimes\nu_{W,V})$). Then there exists $M\in\mathcal{O}(\hat{\mathbf{m}}_{U,W})$ ($M\in\mathcal{O}(\mu_{U,W})$), such that for all $s\notin M$ we have $\hat{\mathbf{n}}_{W,V}(H^s)=0$ ($\nu_{W,V}(H^s)=0$).*

Let $\mathbf{m}$ and $\mathbf{n}$ be dominated measures and suppose that $\mathbf{f}$ is a $\Delta_{U,W}\otimes\nabla_{W,V}$-integrable function on $S\times T$ and let $\mathbf{g}$ differs from $\mathbf{f}$ only on a $(\widehat{\mathbf{m}\otimes\mathbf{n}})_{U,V}$-null set H. Then by Lemma 4.1 there exists a $\hat{\mathbf{m}}_{U,W}$-null set M, such that for all $s\notin M$ the maps $\mathbf{f}(s,\cdot)$ and $\mathbf{g}(s,\cdot)$ differ only on a $\hat{\mathbf{n}}_{W,V}$-null set. Thus, $\mathbf{f}(s,\cdot)$ is $\nabla_{W,V}$-integrable if and only if $\mathbf{g}(s,\cdot)$ is $\nabla_{W,V}$-integrable and if this is the case, their integrals will be equal.

Now we prove the Fubini-type theorem for bounded functions and dominated bornological product measures.

Theorem 4.2. *Let $\mathbf{m}:\Delta\to L(\mathbf{X},\mathbf{Y})$ and $\mathbf{n}:\nabla\to L(\mathbf{Y},\mathbf{Z})$ be measures dominated by the systems of bornological control measures $\{\mu_{U,W};(U,W)\in\mathcal{U}\times\mathcal{W}\}$ and $\{\nu_{W,V};(W,V)\in\mathcal{W}\times\mathcal{V}\}$, respectively. Let $\mathbf{f}:S\times T\to\mathbf{X}_U$ be a U-bounded $\Delta_{U,W}\otimes\nabla_{W,V}$-measurable (and hence $\Delta_{U,W}\otimes\nabla_{W,V}$-integrable) function. Then for $\hat{\mathbf{m}}_{U,W}$-almost all $s\in S$, the map $\mathbf{f}(s,\cdot)$ is $\nabla_{W,V}$-integrable, the map given by*

$$s\mapsto\int_T \mathbf{f}(s,\cdot)\,\mathrm{d}\mathbf{n}$$

for $\hat{\mathbf{m}}_{U,W}$*-almost all* s *(and defined arbitrarily for other* s*) is* $\Delta_{U,W}$*-integrable and we have*

$$\int_{S\times T} \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) = \int_S \int_T \mathbf{f}(s,\cdot)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{m}(s).$$

Proof. From integrability of $\mathbf{f}$ there exists a net $(\mathbf{f}_j)_{j\in J}$ of $\Delta_{U,W}\otimes\nabla_{W,V}$-simple functions converging $(\widehat{\mathbf{m}\otimes\mathbf{n}})_{U,V}$-a.e. to $\mathbf{f}$ on $S\times T$ and for each $G\in\Sigma\otimes\Xi$ holds

$$\lim_{j\in J} p_V\left(\int_G \mathbf{f}\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}) - \int_G \mathbf{f}_j\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n})\right) = 0.$$

Let H be a $(\widehat{\mathbf{m}\otimes\mathbf{n}})_{U,V}$-null set in $\Sigma\otimes\Xi$, such that the sequence $(\mathbf{f}_j)_{j\in J}$ of functions pointwisely converges to $\mathbf{f}$ outside H. Then by Lemma 4.1 there exists a $\hat{\mathbf{m}}_{U,W}$-null set M, such that for all $s\notin M$ we have $\hat{\mathbf{n}}_{W,V}(H^s)=0$. If $s\notin M$, then $(\mathbf{f}_j(s,\cdot))_{j\in J}$ pointwisely converges to $\mathbf{f}(s,\cdot)$ on the complement of H^s. For each $s\in S$ the functions $\mathbf{f}_j(s,\cdot)$, $j\in J$, are $\nabla_{W,V}$-integrable on T, and for each $j\in J$ the map

$$\mathbf{g}_j : s\mapsto \mathbf{f}_j(s,\cdot)$$

is a $\Delta_{U,W}$-simple function on S with values in the space of $\nabla_{W,V}$-simple functions on T.

If $s\notin M$, then $\mathbf{f}_j(s,t)\to\mathbf{f}(s,t)$ for $\hat{\mathbf{n}}_{W,V}$-almost all $t\in T$. Therefore $\mathbf{f}(s,\cdot)$ is $\nabla_{W,V}$-measurable and $\nabla_{W,V}$-integrable and $\int_F \mathbf{f}_j(s,\cdot)\,\mathrm{d}\mathbf{n}\to\int_F \mathbf{f}(s,\cdot)\,\mathrm{d}\mathbf{n}$ for all $s\notin M$ and all $F\in\Xi$. Finally, note that the map

$$\mathbf{h}_j : s\mapsto \int_T \mathbf{f}_j(s,\cdot)\mathrm{d}\mathbf{n}$$

is a $\Delta_{U,W}$-simple function on S with values in $\mathbf{Y}_W$. For all $s\notin M$ the net $(\mathbf{h}_j)_{j\in J}$ of functions pointwisely converges to the map $\mathbf{h}$,

$$\mathbf{h}(s) = \int_T \mathbf{f}(s,\cdot)\,\mathrm{d}\mathbf{n},$$

therefore $\mathbf{h}$ is $\Delta_{U,W}$-measurable and $\Delta_{U,W}$-integrable. Further, according to the bounded convergence theorem we have

$$\lim_{j\in I} p_V\left(\int_S\int_T \mathbf{f}_j(s,\cdot)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{m}(s) - \int_S\int_T \mathbf{f}(s,\cdot)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{m}(s)\right) = 0.$$

Since $\mathbf{f}_j$, $j\in J$, are $\Delta_{U,W}\otimes\nabla_{W,V}$-simple functions, then

$$\int_S\int_T \mathbf{f}_j(s,\cdot)\,\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{m}(s) = \int_{S\times T}\mathbf{f}_j\,\mathrm{d}(\mathbf{m}\otimes\mathbf{n}),$$

which completes the proof. □

References

[1] R. G. Bartle, *A general bilinear vector integral,* Studia Math. **15** (1956), 337–352.

[2] M. E. Ballvé and P. Jiménez Guerra, *Fubini theorems for bornological measures,* Math. Slovaca **43** (1993), 137–148.

[3] N. Dinculeanu, *Vector measures,* Pergamon Press, New York, 1967.

[4] I. Dobrakov, *On integration in Banach spaces, I,* Czechoslovak Math. J. **20** (1970), 511–536.

[5] I. Dobrakov, *On integration in Banach spaces, II,* Czechoslovak Math. J. **20** (1970), 680–695.

[6] I. Dobrakov, *On integration in Banach spaces, III,* Czechoslovak Math. J. **29** (1979), 478–499.

[7] I. Dobrakov, *On representation of linear operators on $C_0(T;X)$,* Czechoslovak Math. J. **21** (1971), 13–30.

[8] I. Dobrakov and J. Farková, *On submeasures II,* Math. Slovaca **30** (1980), 65–81.

[9] M. Duchoň, *A dominancy of vector-valued measures,* Bull. Pol. Sci. **19** (1971), 1085–1091.

[10] M. Duchoň, *Product of dominated vector measures,* Math. Slovaca **27** (1977), 293–301.

[11] N. Dunford and J. Schwartz, *Linear operators, part I,* Interscience Publishers, New York, 1958.

[12] F. J. Fernandez, *On the product of operator valued measures,* Czechoslovak Math. J. **40** (1990), 543 – 562.

[13] P. R. Halmos, *Measure Theory,* Van Nostrand, New York, 1950.

[14] J. Haluška, *On lattices of set functions in complete bornological locally convex spaces,* Simon Stevin **67** (1993), 27–48.

[15] J. Haluška, *On a lattice structure of operator spaces in complete bornological locally convex spaces,* Tatra Mt. Math. Publ. **2** (1993), 143–147.

[16] J. Haluška, *On convergences of functions in complete bornological locally convex spaces,* Rev. Roumaine Math. Pures Appl. **38** (1993), 327–337.

[17] J. Haluška, *On integration in complete bornological locally convex spaces,* Czechoslovak Math. J. **47** (1997), 205–219.

[18] J. Haluška and O. Hutník, *On integrable functions in complete bornological locally convex spaces* (submitted).

[19] J. Haluška and O. Hutník, *On vector integral inequalities,* Mediterr. J. Math. **6**(1) (2009), 105–124.

[20] J. Haluška and O. Hutník, *The Fubini theorem for bornological product measures,* Results Math. (to appear).

[21] J. Haluška and O. Hutník, *The general Fubini theorem in complete bornological locally convex spaces* (preprint).

[22] H. Hogbe-Nlend, *Bornologies and Functional Analysis,* North-Holland, Amsterdam–New York–Oxford, 1977.

[23] H. Jarchow, *Locally Convex Spaces,* Teubner, Stuttgart, 1981.

[24] J. V. Radyno, *Linear Equations and the Bornology* (in Russian), Izd. Bel. Gosud. Univ., Minsk, 1982.

[25] R. Rao Chivukula and A. S. Sastry, *Product vector measures via Bartle integrals,* J. Math. Anal. Appl. **96** (1983), 180–195.

[26] M. M. Rao, *Domination problem for vector measures and applications to nonstationary processes,* In: Springer Lecture Notes in Math. **945**, Springer Verlag, New York, 1982, 296–313.

[27] C. Swartz, *A generalization of a Theorem of Duchoň on products of vector measures,* J. Math. Anal. Appl. **51** (1975), 621–628.

[28] N.-Ch. Wong, *The triangle of operators, topologies and bornologies* (English summary), Third International Congress of Chinese Mathematicians. Part 1, 2, 395–421, AMS/IP Stud. Adv. Math. **42**, pt.1, 2, Amer. Math. Soc., Providence, RI, 2008.

Ján Haluška
Mathematical Institute
Slovak Academy of Science
Grešákova 6
040 01 Košice
Slovakia
e-mail: jhaluska@saske.sk

Ondrej Hutník
Institute of Mathematics
Faculty of Science
P. J. Šafárik University
Jesenná 5
040 01 Košice
Slovakia
e-mail: ondrej.hutnik@upjs.sk

Received: February 27, 2009.
Revised: May 18, 2009.
Accepted: July 7, 2009.