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ON GENERALIZED WEIGHTED QUASI–ARITHMETIC MEANS IN INTEGRAL FORM

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A class of generalized weighted quasi-arithmetic means in the sense of Qi, [6], is studied. In particular, a weighted integral form of Jensen's inequality has as consequences various inequalities and monotonicity properties for the generalized weighted quasi-arithmetic mean $M_{[x,y],g}(p,f)$, $x<y$, with respect to the properties of functions f, g, p.

K e y w o r d s: Jensen's inequality, generalized weighted quasi-arithmetic means, convex function

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1 INTRODUCTION

In the discrete case, the well-known *arithmetic*, *geometric* and *harmonic* means in the non-weighted forms are defined as follows:

$$\frac{1}{n}\sum_{i=1}^{n}a_i, \qquad \left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}, \qquad \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{a_i}}, \tag{1.1}$$

respectively, where $a=\{a_i\}_{i=1}^{n}$ is a sequence of non-negative real numbers. If the following integrals exist, then continuous analogues of the means (1.1) are:

$$\begin{aligned} A_{[x,y]}(f) &= \frac{1}{y-x}\int_x^y f(t)\mathrm{d}t\,,\\ G_{[x,y]}(f) &= \exp\left(\frac{\int_x^y \ln f(t)\,\mathrm{d}t}{y-x}\right),\\ H_{[x,y]}(f) &= \frac{y-x}{\int_x^y \frac{1}{f(t)}\,\mathrm{d}t}, \end{aligned} \tag{1.2}$$

where f and $1/f$ are non-negative real functions Lebesgue integrable on a finite interval $[x,y]\subset\mathbb{R}$ (the set of all real numbers), $x<y$, such that $0<\int_x^y \frac{1}{f(t)}\mathrm{d}t<\infty$. Both of these classical forms of non-weighted means, discrete and continuous, have been generalized, extended, and refined in many various ways [1]. According to the results from 1930's by Kolmogoroff, Nagumo and de Finetti, all types of the so called *intrinsic means* can be expressed via

$$\mathcal{M}_F(\varphi)=\varphi^{-1}\left(\int_{\mathbb{R}}\varphi(t)\mathrm{d}F(t)\right), \tag{1.3}$$

where $F(t)$ is a distribution and $\varphi(t)$ is a continuous real increasing function on $\mathbb{R}$ [2], and the integral (1.3) is meant in the sense of Lebesgue-Stieltjes. This form of means coincides with the so-called *integral* φ*-means.*

Other well-known types of means can be rewritten according to the pattern given by the formula (1.3) as well. For instance, the *generalized logarithmic means* or *Stolarsky means* [7]:

$$\mathcal{M}_F(\varphi)=\mathcal{M}(r;[x,y])=\left(\frac{1}{y-x}\int_x^y t^r\,\mathrm{d}t\right)^{\frac{1}{r}}, \tag{1.4}$$

i.e. $\varphi(t)=t^r$, $r\neq 0$, $r\in\mathbb{R}$. In this paper, we will use the quasi-arithmetic non-symmetrical weighted mean in the sense of Qi, c.f. [6]. In Section 2, we will give a simple proof of Jensen's-type inequality for the weighted integral means. This enables us to derive various inequalities and monotonicity properties for the generalized weighted quasi-arithmetic mean $M_{[x,y],g}(p,f)$, $x<y$, with respect to the properties of functions g, f, p.

2 PRELIMINARIES

Let us denote by $L_1([x,y])$ the vector space of all real integrable functions defined on the interval $[x,y]\subset\mathbb{R}$, $x<y$. Denote by $L_1^+([x,y])=\{p\in L_1([x,y]); p(t)>0, t\in[x,y], 0<\int_x^y p(t)\,\mathrm{d}t=\|p\|_{[x,y]}<\infty\}$.

Let $F\colon\mathbb{R}\to[0,1]$ be a distribution. Let $g\colon[0,\infty]\to\mathbb{R}$ be a real continuous monotone function and $g^{-1}\colon\mathbb{R}\to[0,\infty]$ be the inverse function of g. The *generalized weighted mean of a function* $f\colon[x,y]\to[0,\infty]$ *with respect to the weight function* $p\in L_1^+([x,y])$ is defined as

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follows:

$$\mathcal{M}_{F,g}(p,f) = g^{-1}\left(\int_{\mathbb{R}} p(t)g(f(t))\,\mathrm{d}F(t)\right). \qquad (2.1)$$

In particular, if we consider the distribution

$$F(t) = \begin{cases} 0, & t < x \\ t/\|p\|_{[x,y]}, & x \le t \le y. \\ 1 & y < t \end{cases}$$

and (for the sake of symmetry) $f \in L_1^+([x,y])$, then we obtain the following definition of the generalized weighted mean in the integral form.

Definition 2.1. Let $(p,f) \in L_1^+([x,y]) \times L_1^+([x,y])$. Let $g : [0,\infty] \to \mathbb{R}$ be a real continuous monotone function. The *generalized weighted quasi-arithmetic mean of f with respect to weight function p* is a number $M_{[x,y],g}(p,f) \in \mathbb{R}$, where

$$M_{[x,y],g}(p,f) = g^{-1}\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(f(t))\,\mathrm{d}t\right), \qquad (2.2)$$

where g^{-1} denotes the inverse function of the function g.

Remark 2.2. (a) F. Qi, c.f. [6], Definition 2.1, considered the function f such that $g \circ f \in L_1([x,y])$.

(b) For every $g\colon Im(f) \to \mathbb{R}$,

$$\inf_{t\in[x,y]} f(t) \le M_{[x,y],g}(p,f) \le \sup_{t\in[x,y]} f(t).$$

For the proof see [6, Lemma 2.1].

(c) Many known means in the integral form of two variables p, f are covered in Definition 2.1 when taking the suitable functions g. For instance,

(i) for $g^{-1}(t) = e^t$ put $G_{[x,y]}(p,f) = M_{[x,y],g}(p,f)$, the *generalized weighted geometric mean*;

(ii) for $g^{-1}(t) = t$ put $A_{[x,y]}(p,f) = M_{[x,y],g}(p,f)$, the *generalized weighted arithmetic mean*;

(iii) for $g^{-1}(t) = \frac{1}{t}$ put $H_{[x,y]}(p,f) = M_{[x,y],g}(p,f)$, the *generalized weighted harmonic mean.*

Means $M_{[x,y],g}(p,f)$ generalize also other types of means: logarithmic mean, identric mean, power mean, abstracted mean, one-parameter means, extended logarithmic means, extended mean values, generalized weighted mean values and others. For further reading see [6].

Many mathematical investigations deal with problems about operator of means depending on the behavior of the input functions g, p, f. Rather the most known class of functions is a class of the Jensen convex functions, which originally deal with the arithmetic mean.

Now we extend the Jensen inequality to the class of generalized weighted quasi-arithmetic means in the sense of Definition 2.1.

Lemma 2.3. *Let $(p,f) \in L_1^+([x,y]) \times L_1^+([x,y])$ and let $Im(f) = [\alpha,\beta]$, where $-\infty < \alpha < \beta < \infty$. Let $g\colon [\alpha,\beta] \to \mathbb{R}$.*

(i) *If g is a convex function on $[\alpha,\beta]$, then*

$$g\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,dt\right) \le \frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(f(t))\,dt.$$

(ii) *If g is a concave function on $[\alpha,\beta]$, then*

$$g\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,dt\right) \ge \frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(f(t))\,dt.$$

Proof. We will prove the item (i) concerning the convex function g. The item (ii) can be proved similarly. Put

$$\xi = \frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,\mathrm{d}t. \qquad (2.3)$$

From the mean value theorem of the integral calculus it follows that $\alpha \le \xi \le \beta$. Put

$$\eta = \sup_{\tau\in(\alpha,\beta)} \frac{g(\xi)-g(\tau)}{\xi-\tau},$$

i.e. the supremum of slopes of secant lines. Since g is convex, the function

$$G_{[x,y]}(\xi,\tau) = \frac{g(\xi)-g(\tau)}{\xi-\tau}, \tau \in (\alpha,\beta), \tau \ne \xi,$$

is non-decreasing. Therefore,

$$g(\xi) - g(\tau) \le \eta(\xi - \tau).$$

Choosing, in particular, $\tau = f(t)$ and multiplying both sides of the above equation by $p(t)/\|p\|_{[x,y]}$, we obtain

$$\frac{p(t)}{\|p\|_{[x,y]}}(g(\xi) - g(f(t))) \le \eta\cdot\frac{p(t)}{\|p\|_{[x,y]}}(\xi - f(t)). \qquad (2.4)$$

Integrating of equation (2.4) with respect to t, we have

$$g(\xi) - \frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(f(t))\,\mathrm{d}t \le 0.$$

Indeed,

$$\int_x^y \eta\cdot\frac{p(t)}{\|p\|_{[x,y]}}(\xi - f(t))\,\mathrm{d}t = \frac{\eta}{\|p\|_{[x,y]}}\left(\xi\|p\|_{[x,y]} - \int_x^y p(t)f(t)\,\mathrm{d}t\right) = \eta\left(\xi - \frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,\mathrm{d}t\right) = 0.$$

Replacing ξ by (2.3), we have

$$g\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,\mathrm{d}t\right) \le \frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(f(t))\,\mathrm{d}t.$$

Some well-known inequalities between weighted means in integral form can be obtained as direct corollaries of the Jensen's inequality.

Indeed, put $g^{-1}(t) = \exp(t)$ in Definition 2.1. By applying Lemma 2.3 we directly obtain the integral Arithmetic-Geometric mean inequality $G_{[x,y]}(p,f) \le A_{[x,y]}(p,f)$. This inequality is a special case of the following important inequality between weighted power means.

Corollary 2.4. *Let* $(p,f)\in L_1^+([x,y])\times L_1^+([x,y])$. *Let* $a,b\in\mathbb{R}$ *such that* $a\le b$, *and*

$$P^{[r]}_{[x,y]}(p,f)=\begin{cases}\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f^r(t)\,dt\right)^{\frac{1}{r}}, & r\neq 0\\ \exp\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)\ln f(t)\,dt\right), & r=0.\end{cases}$$

Then

$$P^{[a]}_{[x,y]}(p,f)\le P^{[b]}_{[x,y]}(p,f)\,.$$

This result and some its refinements can be found in [1].

3 ELEMENTARY PROPERTIES OF THE GENERALIZED WEIGHTED QUASI–ARITHMETIC MEANS

Now we will study properties of the class of generalized weighted quasi-arithmetic means expressed in the integral form. An inspiration to study the following inequalities was the article of J. Pečarić, J. Šunde and S. Varošanec, [5], where authors studied (inter alia) some inequalities among quasi-arithmetic means in discrete case.

Lemma 3.1. *Let* $(p,k)\in L_1^+([x,y])\times L_1^+([x,y])$ *and let* $h_i\in L_1^+([x,y])$ *be a sequence of functions,* $i=1,\dots,n$; $n\in\mathbb{N}$. *Let* $\alpha\in\mathbb{R}$.

(a) If g *is a convex function and* f *is a concave function on* $[x,y]$, *then*

- (i) $M_{[x,y],g}(p,k)\ge M_{(-g)}(p,k)$ *and* $M_{[x,y],f}(p,k)\le M_{(-f)}(p,k)$;
- (ii) $M_{[x,y],f}(p,k)\le M_{[x,y],g}(p,k)$;
- (iii) $M_{[x,y],f}(p,\alpha)\le\alpha\le M_{[x,y],g}(p,\alpha)$, *where* $\alpha=\alpha(t)$ *is a constant function for* $t\in[x,y]$;
- (iv) $\sum_{i=1}^n M_{[x,y],f}(p,h_i)\le M_{[x,y],g}\left(p,\sum_{i=1}^n h_i\right)$;
- (v) *if* $f(t)\ge g(t)$ *for all* $t\in[x,y]$, *then* $M_{[x,y],f}(p,g)\le A_{[x,y]}(p,g)\le A_{[x,y]}(p,f)\le M_{[x,y],g}(p,f)$.

(b) If g *is a concave function and* f *is a convex function on* $[x,y]$, *then*

- (i) $M_{[x,y],g}(p,k)\le M_{(-g)}(p,k)$ *and* $M_{[x,y],f}(p,k)\ge M_{(-f)}(p,k)$;
- (ii) $M_{[x,y],f}(p,k)\ge M_{[x,y],g}(p,k)$;
- (iii) $M_{[x,y],f}(p,\alpha)\ge\alpha\ge M_{[x,y],g}(p,\alpha)$, *where* $\alpha=\alpha(t)$ *is a constant function for* $t\in[x,y]$;
- (iv) $\sum_{i=1}^n M_{[x,y],f}(p,h_i)\ge M_{[x,y],g}\left(p,\sum_{i=1}^n h_i\right)$;
- (v) *if* $g(t)\ge f(t)$ *for all* $t\in[x,y]$, *then* $M_{[x,y],f}(p,g)\ge A_{[x,y]}(p,g)\ge A_{[x,y]}(p,f)\ge M_{[x,y],g}(p,f)$.

P r o o f . We will prove (a), the item (b) can be proved analogously. Property (i) follows immediately from (ii) and from standard arguments for convex and concave functions (if g is a convex function, then $-g$ is a concave function, and vice versa). Property (ii) is an easy corollary of (iv) and property (iii) is an easy application of Jensen's inequality for convex and concave functions. Therefore, it remains to show only the properties (iv) and (v).

(iv) Suppose that g is a convex function and f is a concave function. Then by the use of Jensen's inequality, we have

$$M_{[x,y],g}\left(p,\sum_{i=1}^n h_i\right)\ge\sum_{i=1}^n\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)h_i(t)\,\mathrm{d}t\,.$$

Since f is a concave function,

$$\sum_{i=1}^n\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)h_i(t)\,\mathrm{d}t\ge\sum_{i=1}^n M_{[x,y],f}(p,h_i)\,.$$

(v) Let g be a convex function and f be a concave function such that $f(t)\ge g(t)$ for every $t\in[x,y]$. Then a direct calculation yields

$$\begin{aligned}M_{[x,y],g}(p,f)&=g^{-1}\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(f(t))\,\mathrm{d}t\right)\\&\ge\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,\mathrm{d}t\\&=A_{[x,y]}(p,f)\ge A_{[x,y]}(p,g)\\&=\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)g(t)\,\mathrm{d}t\\&\ge f^{-1}\left(\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(g(t))\,\mathrm{d}t\right)\\&=M_{[x,y],f}(p,g).\end{aligned}$$

Theorem 3.2. *Let* $p\in L_1^+([x,y])$ *and* $f\in L_1^+([x,y])$ *be a continuous and integrable function with the continuous first derivative on* (x,y).

(a) *If* f *is a strictly monotone convex function on* $[x,y]$, *then*

$$\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,dt\le f\left(\frac{\int_x^y p(t)f'(t)t\,dt}{\int_x^y p(t)f'(t)\,dt}\right),$$

i.e. $A_{[x,y]}(p,f)\le f\left(A_{[x,y]}(p(t)f'(t),t)\right)$.

(b) *If* f *is a strictly monotone concave function on* $[x,y]$, *then*

$$\frac{1}{\|p\|_{[x,y]}}\int_x^y p(t)f(t)\,dt\ge f\left(\frac{\int_x^y p(t)f'(t)t\,dt}{\int_x^y p(t)f'(t)\,dt}\right),$$

i.e. $A_{[x,y]}(p,f)\ge f\left(A_{[x,y]}(p(t)f'(t),t)\right)$.

P r o o f . Let us define θ by

$$\theta=\frac{\int_x^y p(t)f'(t)t\,\mathrm{d}t}{\int_x^y p(t)f'(t)\,\mathrm{d}t}. \tag{3.1}$$

Since f is strictly monotone on $[x,y]$, $\theta\in(x,y)$. The convexity of f ensures that f' is non-decreasing on (x,y) and

$$f(t)+f'(t)(\theta-t)\le f(\theta), \tag{3.2}$$

c.f. [4]. Multiplying both sides of the inequality (3.2) by $p(t)/||p||_{[x,y]}$, we have

$$\frac{p(t)f(t)}{||p||_{[x,y]}} + \frac{p(t)f'(t)(\theta - t)}{||p||_{[x,y]}} \le \frac{p(t)f(\theta)}{||p||_{[x,y]}}.$$

Integrating the above equation with respect to t we can write

$$\frac{\int_x^y p(t)f(t)\,\mathrm{d}t}{||p||_{[x,y]}} + \theta \cdot \frac{\int_x^y p(t)f'(t)\,\mathrm{d}t}{||p||_{[x,y]}} - \frac{\int_x^y p(t)f'(t)t\,\mathrm{d}t}{||p||_{[x,y]}} \le f(\theta).$$

Replacing θ by (3.1) we obtain

$$\frac{\int_x^y p(t)f(t)\,\mathrm{d}t}{||p||_{[x,y]}} \le f\left(\frac{\int_x^y p(t)f'(t)t\,\mathrm{d}t}{\int_x^y p(t)f'(t)\,\mathrm{d}t}\right).$$

Example 3.3. Suppose that $p \in L_1^+((0,1/2])$ and $f(t) = \ln \frac{1-t}{t}$ on $(0,1/2]$. It is easy to verify that function $f(t)$ is strictly decreasing and convex on $(0,1/2]$ and $f'(t) = \frac{1}{t(1-t)}$, i.e. it satisfies the assumptions of Theorem 3.2. Therefore, we get

$$\frac{1}{||p||_{(0,1/2]}} \int_0^{1/2} p(t) \ln \frac{1-t}{t}\,\mathrm{d}t \le \ln\left(\frac{\int_0^{1/2} \frac{p(t)}{t(1-t)}\,\mathrm{d}t - \int_0^{1/2} \frac{p(t)}{1-t}\,\mathrm{d}t}{\int_0^{1/2} \frac{p(t)}{1-t}\,\mathrm{d}t}\right),$$

which is equivalent to

$$\exp\left(\frac{1}{||p||_{(0,1/2]}} \int_0^{1/2} p(t)\left[\ln \frac{1-t}{t}\right]\,\mathrm{d}t\right) \le \frac{\int_0^{1/2} \frac{p(t)}{t}\,\mathrm{d}t}{\int_0^{1/2} \frac{p(t)}{1-t}\,\mathrm{d}t}.$$

Using notation

$$G_{(0,1/2]}(p,f) = \exp\left(\frac{1}{||p||_{(0,1/2]}} \int_0^{1/2} p(t) \ln f(t)\,\mathrm{d}t\right),$$

and

$$H_{(0,1/2]}(p,f) = \frac{||p||_{(0,1/2]}}{\int_0^{1/2} \frac{p(t)}{f(t)}\,\mathrm{d}t},$$

we may rewrite the last inequality as follows

$$\frac{G_{(0,1/2]}(p(t),1-t)}{G_{(0,1/2]}(p(t),t)} \le \frac{H_{(0,1/2]}(p(t),1-t)}{H_{(0,1/2]}(p(t),t)},$$

which is equivalent to the weighted integral inequality of Wang-Wang, cf. [3], in the form

$$\frac{H_{(0,1/2]}(p(t),t)}{H_{(0,1/2]}(p(t),1-t)} \le \frac{G_{(0,1/2]}(p(t),t)}{G_{(0,1/2]}(p(t),1-t)}.$$

The following theorem is easy and therefore the proof may be omitted.

Theorem 3.4. *Let $p \in L_1^+([x,y])$. Let f and g be two non-negative, continuous and integrable functions on $[x,y]$ such that f has a continuous first derivative on (x,y).*

(a) *If g is convex on $Im(f) = [\alpha,\beta]$, then*

$$A_{[x,y]}(p,f) \le f(x) + M_{[x,y],g}\left(p(t), \int_x^t f'(z)\,dz\right).$$

(b) *If g is concave on $Im(f) = [\alpha,\beta]$, then*

$$A_{[x,y]}(p,f) \ge f(x) + M_{[x,y],g}\left(p(t), \int_x^t f'(z)\,dz\right).$$

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