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# BISCALE OF JOSEF RUT

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## 1. Introduction

Mathematicians and musicians, too, have solved problems connected with various tone systems since antiquity. Connections between mathematics and music are mutually enriching, depending on the progress in mathematics or music. Therefore problems having their origin in music attract the interest of scientists and musicians throughout the history up to present days.

*Pythagorean Tuning*, c.f. [1], was created as a sequence of ratios, i.e. products $2^p 3^q$, where $p$ and $q$ are integers, c.f. Table **??** (column 1). This tuning was established about five hundred years B. C. and used in the Western music up to the 14th century.

The tone system called *Just Intonation* is based on the intervals which are integer exponents of numbers 2, 3, 5, and 7. The key figures in studies of Just Intonation were Gioseffo Zarlino (1517 – 1590), Simon Stevin (1548 – 1620) and Johann Kepler (1571 – 1630). In what follows we will deal with Just Intonation (JI) given by Table **??** (column 2), cf. [2].

The further progress in tuning continued by so-called *temperaments*, cf. [4] which did not avoid the inharmonic music intervals given by irrational numbers. Rather the most popular today, *Equal Temperament*, was already known to Andreas Werckmeister in 1698. It is represented by the sequence

$$W = \{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\} = \{C^{(W)}, C_\sharp^{(W)} = D_\flat^{(W)}, \ldots, B^{(W)}, C'^{(W)}\}.$$

Then there have been many theoretical and experimental works concerned with the assumed universality of *diatonic scales* – both *pro* and *contra*, c.f. [1], [3].

¿From the present musicians' side, something new brings the theory of J. Rut, [5], which the author calls "the relativistic theory of music motion". The result of this theory is the so called *biscale*. Josef Rut introduced the following two versions of 12-tone biscales:

$$c, d, e, f\sharp, g, a, h, c - -c, h\flat, a\flat, g\flat, f, e\flat, d\flat, c \tag{1}$$

and

$$f\sharp, g, a, h, c, d, e, f\sharp - -g\flat, f, e\flat, d\flat, c, h\flat, a\flat, g\flat \tag{2}$$

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| | | | | | |
|---|---|---|---|---|---|
| $C^{(P)}$ | $2^0 3^0$ | 1/1 | $C^{(JI)}$ | $2^0 3^0 5^0$ | 1/1 |
| $D^{(P)}$ | $2^{-3} 3^2$ | 9/8 | $D^{(JI)}$ | $2^1 3^{-2} 5^1$<br>$2^{-3} 3^2$<br>$2^3 7^{-1}$ | 10/9<br>9/8<br>8/7 |
| $E^{(P)}$ | $2^{-6} 3^4$ | 81/64 | $E^{(JI)}$ | $2^{-2} 5^1$ | 5/4 |
| $F_\sharp^{(P)}$ | $2^{-9} 3^6$ | 729/512 | $F_\sharp^{(JI)}$ | $2^{-5} 3^2 5^1$ | 45/32 |
| $G^{(P)}$ | $2^{-1} 3^1$ | 3/2 | $G^{(JI)}$ | $2^{-1} 3^1$ | 3/2 |
| $A^{(P)}$ | $3^3 2^{-4}$ | 27/16 | $A^{(JI)}$ | $3^{-1} 5^1$ | 5/3 |
| $B^{(P)}$ | $2^{-7} 3^5$ | 243/128 | $B^{(JI)}$ | $2^{-3} 3^1 5^1$ | 15/8 |
| $C'^{(P)}$ | $2^1 3^0$ | 2/1 | $C'^{(JI)}$ | $2^1 3^0 5^0$ | 2/1 |
| $C'^{(P)}$ | $2^1 3^0$ | 2/1 | $C'^{(JI)}$ | $2^1 3^0 5^0$ | 2/1 |
| $B_\flat^{(P)}$ | $2^4 3^{-2}$ | 16/9 | $D_\flat^{(JI)}$ | $7^1 2^{-2}$<br>$2^4 3^{-2}$<br>$3^2 5^{-1}$ | 7/4<br>16/9<br>9/5 |
| $A_\flat^{(P)}$ | $2^7 3^{-4}$ | 128/81 | $A_\flat^{(JI)}$ | $2^3 5^{-1}$ | 8/5 |
| $G_\flat^{(P)}$ | $2^{10} 3^{-6}$ | 1024/729 | $G_\flat^{(JI)}$ | $2^6 3^{-2} 5^{-1}$ | 64/45 |
| $F^{(P)}$ | $2^2 3^{-1}$ | 4/3 | $F^{(JI)}$ | $2^2 3^{-1}$ | 4/3 |
| $E_\flat^{(P)}$ | $2^5 3^{-3}$ | 32/27 | $E_\flat^{(JI)}$ | $2^1 3^1 5^{-1}$ | 6/5 |
| $D_\flat^{(P)}$ | $2^8 3^{-5}$ | 256/243 | $D_\flat^{(JI)}$ | $2^4 3^{-1} 5^{-1}$ | 16/15 |
| $C^{(P)}$ | $2^0 3^0$ | 1/1 | $C^{(JI)}$ | $2^0 3^0 5^0$ | 1/1 |

Table 1: Biscale of J. Rut expressed in Pythagorean (P) and Just Intonation (JI) Tuning

which correspond to the major and minor modes, respectively. Observe, that both modes (**??**) and (**??**) have different ascendent and descendent forms (hence the name – biscale). Note that original biscales of J. Rut were derived from tone $d$.

These biscales are not bounded with any special tone system. However, J. Rut dealt only with the ordinar 12-tone Equal Temperament. Even in this case, the musical results (we mean compositions and interpretation) are meaningful. The question which naturally arises is – what will happen when considering another tone system(s) which differs the 12-tone Equal Temperament?

In the present paper, we find the connection among these four tone systems (Rut, 12-tone Equal Temperament, Pythagorean system, and Just Intonation). It would be clear that the investigation is a model of several types of uncertainty (strife, fuzziness, and principal unpreciseness). Two intervals causing a dilemma is a pair $(f\sharp, g\flat)$ (known also as tritone, the *diabolo in musica*).

## 2. Results

Denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ the sets of all natural, integer, rational, and real numbers, respectively. Denote by $\mathcal{L} = ((0,\infty), \cdot, 1, \leq)$ the usual multiplicative group with the natural order on $\mathbb{R}$. So, if $a \leq b$ with $a, b \in (0,\infty)$, then $b/a$ is an $\mathcal{L}$-length of the interval $(a,b)$. Since this terminology is not obvious, we borrow the usual musical terminology, i.e. we simply say that $b/a$ is an interval. This inaccuracy does not lead to any misunderstanding because in this paper the term "interval" is used only in this sense.

The proof of the following lemma is easy.

**Lemma 1** *Let* $q = \text{const} \in (0,\infty)$*. Then* $\rho_q(u,v) = \left|\log_2 \sqrt[q]{u/v}\right|$, $u, v \in (0,\infty)$, *is a metrics on* $\mathcal{L}$.

Now we show that the biscale of J. Rut expressed in Pythagorean values ($R_P$) is closer to Equal Temperament than that biscale expressed in Just Intonation values ($R_{JI}$). As a unit $q$ for imagination of measure values of corresponding intervals in $R_P$ and $R_{JI}$, we take

$$q = \log_2(G^{(W)}/G^{(JI)}) = \log_2 3 - (19/12) \approx 0.001629167$$

for the metrics $\rho_q$, c.f. Lemma 1. Clearly,

$$\rho_q(G^{(W)}, G^{(JI)}) = 1.$$

Note that the equal tempered fifth is the best approximated (to JI) music interval in Equal Temperament (not mentioning trivially the unison and octave).

**Theorem 1** *For each (qualitative music) interval $C, D_\flat, \ldots, B, C'$, but intervals $F_\sharp, G_\flat$, the biscale of Rut (expressed in Pythagorean Tuning) $R_P$ values are closer to Equal Temperament values than the biscale of Rut $R_{JI}$ (expressed in Just Intonation) values in the metrics $\rho_q$ with $q = \log_2 3 - (19/12)$.*

**Proof.** c.f. Table **??** □

We see that the representation of the biscale of J. Rut via Pythagorean System values is closer to the 12-tone Equal Temperament System than the representation of the biscale of J. Rut via Just Intonation in each value but the tritone (ambigously, both $F_\sharp$ and $G_\flat$ ). This assertion is well in accord with the Rut's claims about a specific role of the tritones in his biscale theory and in music in general. To evaluate this discrepancy, now we introduce an aggregation metrics between sets (the Euclidean metrics for the 12 dimensional vector space) and show that (with respect to this set metrics) the distance between the 12-tone Equal Temperament and $R_P$ is less than the distance between the 12-tone Equal Temperament and $R_{JI}$.

**Lemma 2** *The following function*

$$\rho(R_P, W) = \sqrt{\sum_{u_i \in R_P, v_i \in W} \rho_q(u_i, v_i)^2} \tag{3}$$

*where $u_i \in R_P, v_i \in W$, is a metrics on $\mathcal{L}$ (analogously for $u_i \in R_{JI}, v_i \in W$).*

Now we are able to prove the following main result.

**Theorem 2** *For each (qualitative music) interval $C, D_\flat, \ldots, B, C'$, the $R_P$ values are closer to Equal Temperament values than the $R_{JI}$ values in the Euclidean metrics (**??**).*

**Proof.** It is easy to verify, c.f. Table 2, that

$$\rho(R_P, W) < \rho(R_{JI}, W).$$

□

| | $\rho(R_P, W)$ | $\rho(R_{JI}, W)$ |
|---|---|---|
| $C$ | 0 | 0 |
| $D$ 10/9 | | 9,037986895 |
| $D$ 9/8 | 2,000001316 | 2,000001316 |
| $D$ 8/7 | | 15,94588599 |
| $E$ | 4,000000254 | 7,000656511 |
| $F_\sharp$ | 6,000001847 | 5,000656236 |
| $G$ | 1,000000759 | 1,000000759 |
| $A$ | 3,000001281 | 8,000656408 |
| $B$ | 5,000001477 | 6,000656322 |
| $C'$ | 0 | 0 |
| $B_\flat$ 7/4 | | 15,94582499 |
| $B_\flat$ 16/9 | 2,000000558 | 2,000000558 |
| $B_\flat$ 9/5 | | 9,000657557 |
| $A_\flat$ | 4,000000777 | 7,000657003 |
| $G_\flat$ | 6,000001049 | 5,000656033 |
| $F$ | 1,000000568 | 1,000000568 |
| $E_\flat$ | 3,000000464 | 8,000657188 |
| $D_\flat$ | 5,000001103 | 6,000650906 |
| $C$ | 0 | 0 |

Table 2: Pythagorean Tuning ($R_P$) versus Just Intonation ($R_{JI}$)

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## Zusammenfassung


*Wir präsentieren die bi-Skala von J. Rut als Zahlenmengen in der pythagoreischen Stimmung (a) und ebenfalls in der reinen Stimmung (b). Wir zeigen, dass in jedem Wert ausser dem Triton ($f_\sharp, g_\flat$) die Repräsentation (a) zu dem 12-Töne gleichmässig temperierten System näher ist als die Repräsentation (b). Diese Behauptung ist in Übereinstimmung mit den Ansprüchen von Rut an die Funktion des Tritons in seiner Theorie der bi-Skala und in der Musik generell. Um diese Missweisung zu eliminieren, führen wir eine Metrik zwischen zwei Mengen ein und wir zeigen, dass die Distanz zwischen dem 12-Töne gleichmässig temperierten System und (a) kleiner als die Distanz zwischen dem 12-Töne gleichmässig temperierten System und (b) ist.*