Tatra Mt. Math. Publs. 54(2004), 423–432.

A skeleton of Fubini-type theorem in vector spaces for the Kurzweil integral and operator measures

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Abstract. A Fubini theorem in vector spaces for the Kurzweil integral with respect to operator measures is proven.

Keywords: Fubini theorem, Kurzweil integral, operator measures, vector spaces, bornology

AMS Subj. Class.: Primary: 46G10, Secondary: 21B15, 21B05, 28B10

Thanks: This article was supported with the SAS-CNR project (2001 – 2003) "Integration in vector spaces equipped with additional structures" and Project VEGA 7192.

1 Introduction

In [1], [2], [7], there is given a generalization of Kurzweil integration pattern to complete vector lattices, Riesz spaces, and to compact topological spaces, respectively. Concerning the Fubini theorem, in [4], we generalized the Dobrakov integral to complete bornological locally convex spaces. This generalization involves the Fubini theorem. In [9], the problem of the existence of the product measure in the context of locally convex spaces for bilinear integrals is solved in general.

In this paper, we show a general scheme how to prove a Fubini-type theorem for operator valued measures and Kurzweil-Henstock type integration in vector spaces. We do not touch the problems about the existence of the product measure and partial integrals. They are solved in the classical case but, in the operator measure setting, there are known only some special results. Moreover, we do not consider the problem of measurability of integrable functions in the general context.

2 Preliminaries

2.1 Construction of the integral

To recall the construction of the Kurzweil-Henstock integral, cf. [8].

The following definition introduces the set structure on the domain of functions which we integrate, the set T.

Definition 1 Let $T \neq \emptyset$ be a compact topological space. Let U be the class of all functions $u : T \to 2^T$ such that for every $t \in T$, $u(t)$ is an open neighbourhood of the point t. Denote by B the σ-algebra of all Borel subsets of T. We say that π is a *partition* of the set T if

$$\pi = \{(E^{(i)}, t^{(i)}); t^{(i)} \in \overline{E^{(i)}}, \overline{E^{(i)}} \in B, i = 1, 2, \ldots, I\},$$

where the sets $E^{(i)}, i = 1, 2, \ldots, I$, are pairwise disjoint, the union of them is the whole set T, and $\overline{E}$ is the closure of E in the topology of the space T. By A we denote the *class of all partitions* π of the set T such that $E^{(i)} \subset u(t^{(i)}), u \in U, i = 1, 2, \ldots, I$.

For T is compact, $A(u) \neq \emptyset$ for every $u \in U$ (the Cousin lemma, cf. [5], Proposition 5.1.8., pp. 73 – 74).

To construct the integral, we will consider the following construction of measure.

Let X, Y be two real or complex vector spaces. Let $L(X, Y)$ be the space of all linear operators $L : X \to Y$. Let $D_X \neq \emptyset$, $D_Y \neq \emptyset$ be two lattices of Banach disks such that $\bigcup_{D \in D_X} D = X$ and $\bigcup_{D \in D_Y} D = Y$.

Example 1 The simplest situation occurs, when X and Y are Banach spaces over the field $\mathcal{K}$ where $\mathcal{K} = \mathcal{R}$ (the field of all real numbers) or $\mathcal{C}$ (the field of all complex numbers). In this case, $D_X = \{\lambda H^*; \lambda \in \mathcal{K}\}$ and H^* may be taken as only one Banach disk. Similarly for Y. In the setting of this paper, the cardinality of the set of such "generators" H^* is arbitrary (including e.g. the cardinality of the set of all real numbers or of all real functions defined on the interval $[0, 1]$ in particular). Note also, that no order structures are supposed on X or Y.

Example 2 The mentioned general situation, the lattice of Banach disks, can be introduced for every real or complex vector spaces X and Y. Indeed, let $\tilde{D}_X$ be the set of all Banach disks of the vector space X. If $H', H'' \in \tilde{D}_X$, the lattice operation may be defined, e.g., as follows:

$$H' \wedge H'' = H' \cap H'', H' \vee H'' = acs(H' \cup H''), \tag{1}$$

where *acs* denotes the topological closure of the absolutely convex span of the set. For more details, cf. [3], Lemma 1.7.

Example 3 A lattice $\hat{D}_X$ which differs from $\tilde{D}_X$, cf. Example 2. Let $\hat{D}_X$ consist of all Banach disks of finite dimensional vector subspaces of the vector space X. The lattice operations on $\hat{D}_X$ may be given by (1).

Let $Q_X = \{p_{H_1}; H_1 \in D_X\}$ where p_{H_1} is the Minkowski functional of H_1. Similarly, $Q_Y = \{p_{H_2}; H_2 \in D_Y\}$. Minkowski functionals $p_{H_1} \in Q_X, p_{H_2} \in Q_Y$ are norms and the linear spans $X(H_1), Y(H_2)$ are Banach spaces in this case. Note that the trivial case $H_0 = \{0\} \in D_X$ will not deform the theory when putting $p_{H_0}(x) = \infty$ for every $x \in X$ (analogously for Y). We will suppose this everywhere in the ongoing text.

We say that the sequence $L_n \in L(X, Y)$, $n = 1, 2, \ldots$, of operators (D_X, D_Y)-*converges* to the operator $L \in L(X, Y)$ if there exist $H_1 \in D_X, H_2 \in D_Y$, such that $L_n \in L(X, Y)$ converges to the operator $L \in L(X, Y)$ in the strong operator topology for the Banach spaces $X(H_1)$ and $Y(H_2)$. Let the measure $m : B \to L(X, Y)$ be σ-additive with respect to the (D_X, D_Y)-*convergence* on the space $L(X, Y)$.

Now we are able to define the following Kurzweil-Henstock-type integral with respect to the operator valued measure:

Definition 2 Let $H_1 \in D_X$ and $H_2 \in D_Y$. A function $f : T \to X$ is called to be (H_1, H_2)-*integrable* if $f(T) \subset X(H_1)$ and there exists $y \in Y(H_2)$

such that
$$\forall \varepsilon > 0, \exists u \in U, \forall \pi \in A(u),$$
there holds
$$p_{H_2}(S(f,\pi) - y) < \varepsilon,$$
where
$$S(f,\pi) = \Sigma_{i=1}^{I} m(E^{(i)}) f(t^{(i)}).$$
The function $f : T \to X$ is called *integrable* if there exist $H_1 \in D_X$ and $H_2 \in D_Y$ such that f is (H_1, H_2)-integrable. The element $y \in Y$ is called *integral* and written
$$y = \int_T f dm.$$

Remark 1 It can be proved that the value of the integral is unique if it exists, the integral is a linear operator and a finitely additive set function. It is not hard to see that the integral of every simple function exists and, therefore, every simple function is integrable.

2.2 Formulation of the Fubini theorem

To formulate and prove the Fubini theorem, we need to introduce some additional denotations.

Let $T_1 \neq \emptyset$ and $T_2 \neq \emptyset$ be two compact topological spaces. Then $T = T_1 \times T_2$ is a compact topological space, too.

Let B_1 and B_2 be two σ-algebras of all Borel subsets of T_1 and T_2, respectively. Let B denotes the smallest σ-algebra generated by all rectangles of the type $E_1 \times E_2$, where $E_1 \in B_1$ and $E_2 \in B_2$, respectively.

Let $X \neq \emptyset$, $Y \neq \emptyset$, and $Z \neq \emptyset$ be three real or complex vector spaces. Let $D_X \neq \emptyset$, $D_Y \neq \emptyset$, and $D_Z \neq \emptyset$ be three lattices of Banach disks such that $\bigcup_{D \in D_X} D = X, \bigcup_{D \in D_Y} D = Y$, and $\bigcup_{D \in D_Z} D = Z$. Denote by Q_X, Q_Y, Q_Z the systems of all Minkowski functionals corresponding to Banach disks from D_X, D_Y, D_Z, respectively.

Denote by $L(X,Y)$, $L(Y,Z)$, and $L(X,Z)$ the vector spaces of all linear operators acting from $X \to Y$, $Y \to Z$, and $X \to Z$ respectively.

Let $m_1 : B_1 \to L(X,Y)$ and $m_2 : B_2 \to L(Y,Z)$ be two operator valued measures σ-additive with respect to the (D_X, D_Y)- and (D_Y, D_Z)- convergences. We say that the *product measure* $m_1 \otimes m_2$ of measures m_1 and m_2 exists on B if there exists a unique $L(X,Z)$-valued measure m σ-additive in the (D_X, D_Z)-convergence such that for every $E_1 \in B_1$ and $E_2 \in B_2$,
$$m(E_1 \times E_2) = (m_1 \otimes m_2)(E_1 \times E_2) = m_2(E_2) m_1(E_1).$$

For the σ-additive product measures which are σ-additive in operator topologies, cf. [9].

Let the function $f : T = T_1 \times T_2 \to X$ be given. Denote the following functions if they exist in their domains:

$$f_1 : T_1 \times T_2 \to X, f_1(t_1, t_2) = f(t_1, t_2), \quad (2)$$

$$f_2 : T_2 \to Y, f_2(t_2) = \int_{T_1} f_1(\cdot_1, t_2) dm_1, \quad (3)$$

$$f_3 : \int_{T_2} f_2(\cdot_2) dm_2 = \int_{T_2} \left(\int_{T_1} f_1(\cdot_1, \cdot_2) dm_1 \right) dm_2 = \tilde{y} \in Z. \quad (4)$$

If the integral $\tilde{y}$ exists, it is called the *multiple integral.*

If the integral

$$y_{\otimes} = \int_T f \, dm = \int_{T_1 \times T_2} f \, d(m_1 \otimes m_2) \quad (5)$$

exists, it is called it the *product integral.*

Concerning the integral (5), we will use the symbols $\varepsilon_{\otimes}$, $U_{\otimes}$, $\pi_{\otimes}$, $A_{\otimes}$, $S_{\otimes}$ (cf. Definition 1 and Definition 2) in the proof of the Fubini theorem.

Definition 3 By (H_2, H_3)-*semivariation* we mean the function

$$\|m_2\|_{H_2,H_3} : B_2 \to [0, \infty]$$

defined as follows:

$$\|m_2\|_{H_2,H_3}(E_2) = \sup p_{H_3} \left(\sum_{j=1}^{I_2} m_2(E_2^{(j)}) x_2^{(j)} \right) \quad (6)$$

where $H_2 \in D_X, H_3 \in D_Z$ and the supremum is taken over the all finite disjoint partitions $E_2^{(j)} \in B_2, j = 1, 2, \ldots, I_2$, the union of them is E_2 and over all the collections of elements $x_2^{(j)} \in H_2, j = 1, 2, \ldots, I_2$; if $\sup p_{H_3} \left(\sum_{j=1}^{I_2} m_1(E_2^{(j)}) x_2^{(j)} \right)$) does not exist or when $H_2 = \{0\}$ or $H_3 = \{0\}$, we put $\|m_2\|_{H_2,H_3}(E_2) = \infty$.

We say that the measure m_2 is of finite semivariation if there exist $H_2 \in D_X, H_3 \in D_Z$ and $K < \infty$ such that

$$\|m_2\|_{H_2,H_3}(T_2) = K.$$

It is easy to see that $\|m_2\|_{H_2,H_3}$ is a monotone, σ-subadditive set function such that $\|m_2\|_{H_2,H_3}(\emptyset) = 0$, where $H_2 \neq \{0\}, H_3 \neq \{0\}$.

Let us introduce the following seminorm on the set of all functions $f_2 : T_2 \to Y$.

Definition 4 Let $f_2 : T_2 \to Y$. Denote by

$$\|f_2\|_{H_2} = \sup_{t \in T_2} p_{H_2}(f_2(t)),$$

where p_{H_2} denotes the Minkowski functional of the set $H_2 \in D_Y$;
if $\sup_{t\in T_2} p_{H_2}(f_2(t))$ does not exist or $H_2 = \{0\}$, we put $\|f_2\|_{H_2} = \infty$.

By Definition 4, we can reformulate (6) as follows:

$$\|m_2\|_{H_2,H_3}(E_2) = \sup_{\|f\|_{H_2} \le 1} p_{H_3}\left(\int_{T_2} f_2 \cdot \chi_{E_2}\, dm_2\right) = \sup_{\|f\|_{H_2} \le 1} p_{H_3}\left(\int_{E_2} f_2\, dm_2\right)$$

where $E_2 \in B_2$ and $f_2 : T_2 \to Y$ are simple functions.

Immediately, Definition 4 also implies:

Lemma 1 *Let $H_2 \in D_Y, H_3 \in D_Z, H_2 \neq \{0\}, H_3 \neq \{0\}$. For every (H_2, H_3)-integrable function $f_2 : T_2 \to Y$,*

$$p_{H_3}\left(\int_{T_2} f_2 dm_2\right) \le \|f_2\|_{H_2} \cdot \|m_2\|_{H_2,H_3}(T_2).$$

The formulation of the Fubini theorem in its "skeleton" form is as follows.

Theorem 1 *(Fubini) Let X, Y, Z be three real or complex vector spaces equipped with lattices of Banach disks D_X, D_Y, D_Z, such that $\bigcup_{D\in D_X} D = X, \bigcup_{D\in D_Y} D = Y, \bigcup_{D\in D_Z} D = Z$, respectively. Let T_1, T_2 be two compact spaces. Let B be the Borel σ-algebra of subsets of $T_1 \times T_2$. Let $m_1 : B_1 \to L(X,Y), m_2 : B_2 \to L(Y,Z)$ be two operator valued measures σ-additive in the (D_X, D_Y)- and (D_Y, D_Z)- convergences. Let $m_1 \otimes m_2$ be a Borel measure σ-additive in the (D_X, D_Z)- convergence. Let the measure m_2 be of finite semivariation. Let there exist*

1. *the product measure $m_1 \otimes m_2 : B \to L(X,Z)$;*
2. *the product integral $y_\otimes = \int_T f\, dm = \int_{T_1\times T_2} f\, d(m_1 \otimes m_2)$;*
3. *the multiple integral $\tilde{y} = \int_{T_2}\left(\int_{T_1} f(\cdot_1, \cdot_2) dm_1\right) dm_2$.*

Then $y_\otimes = \tilde{y}$.

3 Proof of the theorem

By hypothesis, the integrals $y_\otimes$ and $\tilde{y}$ exist. For $l = 1$ and 2, let ε_l, A_l, U_l, u_l, π_l, S_l, I_l, $E_l^{(i)}$, $i = 1, \ldots, I_l$, be associated to T_l with the same role as ε, A, U, u, π, S, I, $E^{(i)}$, $i = 1, \ldots, I$ in Definitions 1 and 2 relatively to T_1 and T_2, respectively.

Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ be given.

By virtue of the Cousin's lema, there hold

$$A_1(u_1) \neq \emptyset, \quad A_2(u_2) \neq \emptyset.$$

Firstly, consider the integral sums $S_\otimes(f, \pi_\otimes)$ of $y_\otimes$ which are of special form:

$$S_\otimes(f, \pi_\otimes) = \sum_{j=1}^{I_2} \sum_{i=1}^{I_1} m_2(E_2^{(j)})\, m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)}).$$

Let $S_2(f_2, \pi_2) = \sum_{j=1}^{I_2} m_2(E_2^{(j)})\, f_2(t_2^{(j)})$ be the integral sum of $\tilde{y}$.

By Definition 2, there are $H', H'' \in D_Z$ such that $y_\otimes - S_\otimes(f, \pi_\otimes) \in Z(H')$ and $\tilde{y} - S_2(f_2, \pi_2) \in Z(H'')$, respectively. Put $H_3 = H' \vee H'' \in D_Z$. Consider

$$\begin{aligned} & p_{H_3}(S_\otimes(f, \pi_\otimes) - S_2(f_2, \pi_2)) \\ = \; & p_{H_3}\left(\sum_{j=1}^{I_2} \sum_{i=1}^{I_1} m_2(E_2^{(j)})\, m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)}) - \sum_{j=1}^{I_2} m_2(E_2^{(j)})\, f_2(t_2^{(j)}) \right) \\ = \; & p_{H_3}\left(\sum_{j=1}^{I_2} m_2(E_2^{(j)}) \left[\sum_{i=1}^{I_1} m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)}) - f_2(t_2^{(j)}) \right] \right). \end{aligned} \tag{7}$$

The expression

$$\Phi_2(t_2) = \sum_{j=1}^{I_2} \left[\sum_{i=1}^{I_1} m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)}) - f_2(t_2^{(j)}) \right] \chi_{E_2^{(j)}}(t_2)$$

is a simple function, $\Phi_2 : T_2 \to X_2$ (and hence, it is integrable). So, we may apply Lemma 1. To do this, replace the integrals $f_2(t_2^{(j)}) \in Z(H_3)$, $j = 1, 2, \ldots, I_2$, by their integral sums. We continue (7):

$$= \quad p_{H_3}\left(\sum_{j=1}^{I_2} m_2(E_2^{(j)}) \left[\left\{ \sum_{i=1}^{I_1} m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)}) \right.\right.\right.$$

$$
\begin{aligned}
&- \sum_{k_j=1}^{K_{I_2}} m_1(E_1^{(k_j)}) f(t_1^{(k_j)}, t_2^{(j)}) \Bigg\} \\
&+ \left\{ \sum_{k_j=1}^{K_{I_2}} m_1(E_1^{(k_j)}) f(t_1^{(k_j)}, t_2^{(j)}) - \int_{T_1} f(., t_2^{(j)}) dm_1 \right\} \Bigg] \Bigg), \qquad (8)
\end{aligned}
$$

where

$$\sum_{k_j=1}^{K_{I_2}} m_1(E_1^{(k_j)}) f(t_1^{(k_j)}, t_2^{(j)})$$

is an integrable sum of

$$\sum_{i=1}^{I_1} m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)})$$

(it is simple and, hence, integrable). And, by construction,

$$\sum_{k_j=1}^{K_{I_2}} m_1(E_1^{(k_j)}) f(t_1^{(k_j)}, t_2^{(j)})$$

is an integrable sum of

$$f_2(t_2^j) = \int_{T_1} f(._1, t_2^{(j)}) dm_1.$$

The expression

$$p_{H_2} \left(\sum_{k_j=1}^{K_{I_2}} m_1(E_1^{(k_j)}) f(t_1^{(k_j)}, t_2^{(j)}) - \int_{T_1} f(., t_2^{(j)}) dm_1 \right) < \frac{\varepsilon_1}{2}, \ j = 1, 2, \ldots, I_2,$$

by construction.

The expressions

$$p_{H_2} \left(\sum_{i=1}^{I_1} m_1(E_1^{(i)}) f(t_1^{(i)}, t_2^{(j)}) - \sum_{k_j=1}^{K_{I_2}} m_1(E_1^{(k_j)}) f(t_1^{(k_j)}, t_2^{(j)}) \right), \ j = 1, 2, \ldots, I_2,$$

we make less than $\frac{\varepsilon_1}{2}$ using a common refinement of the set T_1. This refinement is possible because of the special form of $S_{\otimes}(f, \pi_{\otimes})$.

Thus, the expression in braketts $[\ldots]$, cf. 8, can be made such that so that $\|[\ldots]\|_{H_2,H_3} \leq \varepsilon_1/2 + \varepsilon_1/2 = \varepsilon_1$.

For $H_2 \in D_Y$ and $H_3 \in D_Z$, and taking specially $\varepsilon_1 \leq 1$, the expression in $[\ldots]$ is a collection of elements $x_2^{(j)} \in H_2, j = 1, 2, \ldots, I_2$, satisfying Definition 3 with respect to the measure m_2. Since the measure m_2 is of finite semivariation by assumption, there exists a constant K, $0 < K < \infty$ such that

$$\|m_2\|_{H_2,H_3}(T_2) = K.$$

Clearly, if $\varepsilon_1 \to 0, 0 < \varepsilon_1 \leq 1$, then

$$\sup_{\|f\|_{H_2} \leq \varepsilon_1 \leq 1} p_{H_3}\left(\int_{T_2} f_2 \; dm_2\right) \leq \|m_2\|_{H_2,H_3}(T_2) = \sup_{\|f\|_{H_2} \leq 1} p_{H_3}\left(\int_{T_2} f_2 \; dm_2\right),$$

where $f_2 : T_2 \to Y$ is a simple function (the first supremum is taken over a subset).

We have:

$$p_{H_3}(S_\otimes(f, \pi_\otimes) - (S_2(f_2, \pi_2)) < \varepsilon_1 K \tag{9}$$

and (9) implies $\varepsilon_1 K > p_{H_3}(S_\otimes(f, \pi_\otimes) - \tilde{y} + \tilde{y} + S_2(f_2, \pi_2))$. Thus,

$$p_{H_3}(S_\otimes(f, \pi_\otimes) - \tilde{y}) < p_{H_3}(S_2(f_2, \pi_2) - \tilde{y}) + \varepsilon_1 K < \varepsilon_2 + \varepsilon_1 K. \tag{10}$$

Denote by $\varepsilon_\otimes = \varepsilon_2 + \varepsilon_1 K$. For every $t_1 \in T_1$, $t_2 \in T_2$, denote by $u_\otimes$ the function associated with $\varepsilon_\otimes$, where $u_\otimes(t_1, t_2) = u_1(t_1) \times u_2(t_2) \in U_\otimes$. Then the partition

$$\pi_\otimes = \{(E^{(i,j)}, (t_1^{(i)}, t_2^{(j)})) : E^{(i,j)} = E_1^{(i)} \times E_2^{(j)},$$
$$(E_1^{(i)}, t_1^{(i)}) \in \pi_1, (E_2^{(j)}, t_2^{(j)}) \in \pi_2, (t_1^{(i)}, t_2^{(j)}) \in E^{(i,j)}\}$$

satisfies the condition

$$E^{(i,j)} \subset u_\otimes(t_1^{(i)}, t_2^{(j)}), \quad i = 1, 2, \ldots, I_1; \quad j = 1, 2, \ldots, I_2.$$

In other words, $S_\otimes(f, \pi_\otimes)$ is an integral sum of the integral y. Since y exists by assumption, (10) and Lemma 1 imply that for every integral sum $S_\otimes(f, \pi)$, where $\pi \in A_\otimes(u_\otimes)$ is now an arbitrary partition, there holds:

$$\begin{aligned} \varepsilon_\otimes &> p_{H_3}(S_\otimes(f, \pi_\otimes) - S_\otimes(f, \pi) + S_\otimes(f, \pi) - \tilde{y}) \qquad (11) \\ &\geq p_{H_3}(S_\otimes(f, \pi_\otimes) - S_\otimes(f, \pi)) - p_{H_3}(S_\otimes(f, \pi) - \tilde{y}), \end{aligned}$$

i.e.,

$$p_{H_3}(S_\otimes(f, \pi) - \tilde{y}) < 2\,\varepsilon_\otimes.$$

Therefore $y_\otimes = \tilde{y}$. $\square$

References

[1] BOCCUTO A., Abstract integration in Riesz spaces, *Tatra Mt. Math. Publs.* **5** (1995), 107 – 124.

[2] HALUŠKA J., On integration in complete vector lattices, *Tatra Mt. Math. Publs.* **3** (1993), 201 – 212.

[3] HALUŠKA J., On lattices of set functions in complete bornological locally convex spaces, *Simon Stevin* **67** (1993), 27 – 48.

[4] HALUŠKA J., On integration in complete bornological locally convex spaces, *Czechoslovak Math. J.* **47** (1997), 205-219.

[5] LEE PENG Y. – VÝBORNÝ R., *The Integral: An Easy Approach after Kurzweil and Henstock,* Cambridge University Press, Cambridge 2000.

[6] KURZWEIL J., *Nicht absolut konvergente Integrale* Teubner Texts in Mathematics, Leipzig 1980.

[7] RIEČAN B., On the Kurzweil integral in compact topological spaces *Radovi Matematički* **2** (1986), 151 – 163.

[8] RIEČAN B. – NEUBRUNN T., *Integral, Measure and Ordering,* Kluwer–Ister Science, Dordrecht – Bratislava, 1997.

[9] RAO CHIVUKULA R. – SASTRY A. S., Product vector measures via Bartle Integrals, *J. of Math. An and Appl.* **96** (1983), 180 – 195.