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Unimodular matrices and diatonic scales

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Abstract

In this paper we will deal with a subset of a group of all unimodular 3×3 matrices (noncommutative group of matrices A, such that $det(A) = 1$) derived from geometrical nets with 3 quotients (bases). The research is inspired with diatonic scales in music.
Keywords: tone system, unimodular matrix, diatonic scale, ambiguity

1 Introduction

Considering the 12-granulation of octave in European music, the ambiguous intervals in the 17-valued Just Intonation Set are the second and the minor seventh and the tritone (the relative frequencies 10/9, 9/8, 8/7; and 7/4, 16/9, 18/10; and 45/32, 64/45, respectively). Pythagorean Tuning we can consider also 17- or more valued. Indian shrutis yield another type of ambiguity – it is present in the form of ragas which are restrictions of 22-tone Pythagorean scale mostly to 7 tone scales. Finally, scales of Pacific region, Slendro and Pelog, represent fuzzy type of uncertainty because there is no crisp defined pitch of tones. In this paper we will study the ambiguity of European tone systems which contain the major scale.

Precise the question as follows: find all 5-limit 12-tone tone systems containing the C-major scale (relative frequencies 1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8, and 2/1).

It is known that each tone system (and scales in particular) can be represented via a geometrical net which is a generalization of the geometrical progression notion. So, let us search the answer to the question in the form of geometrical net. The results will be then consequences of solutions of Diophantine equations describing the basic acoustic relations among octave, perfect fifth and major (minor) third. The 12-granule system has properties of the Just Intonation Set (it involves octave, perfect fifth, perfect fourth, major third, minor third, major whole tone, minor whole tone, diatonic semitone and chromatic semitone) and also of Pythagorean Tuning. Further, there are applications to superparticular ratios.

The ratios $256/243, 25/24, 16/15$ are known as the *minor Pythagorean, chromatic,* and *diatonic semitone*, respectively. Just Intonation [when we avoid the minor seventh (7/4) and the second (8/7)] is constructed on the basis of the chromatic and diatonic semitones. This tone system is often mentioned in Europe as the most natural tuning from many viewpoints (physical, psycho-acoustical, polyphonic, etc.). On the other side, Pythagorean Tuning is based exclusively on the minor Pythagorean semitone, *diesis* (256/243). Thus the Just Intonation Set and Pythagorean Tuning are considered by music theoreticians as two fully incompatible tone systems.

This viewpoint is not correct. In fact, not only the diatonic and chromatic *but also the minor Pythagorean semitone* (together with the diatonic semitone and its complement to the major whole tone) can serve as a basis for the construction of 12-granule diatonic scales, cf. Table 3. Further, we will show in this paper why the gypsy scales are important tone systems.

For the sake of brevity, we will use the following notation of the geometric net (more precisely, the n-quotient geometric net) $\langle \Gamma_i \rangle$ in $\mathcal{R}$:

$$\langle \Gamma_i \rangle = \langle X^{\nu_{i,\cdot}}; |\nu_{i,\cdot}| = i, \ldots \leq \nu_{0,\cdot} \leq \nu_{1,\cdot} \leq \ldots \leq \nu_{i,\cdot} \leq \ldots \rangle_{\nu_{i,\cdot} \in \mathcal{Z}^3},$$

where

$$X = (X_1, X_2, \cdots, X_n) \in \mathcal{R}^n, \nu_{i,\cdot} = (\nu_{i,1}, \nu_{i,2}, \ldots, \nu_{i,n}) \in \mathcal{Z}^n,$$

$$\nu_{i,\cdot} \leq \nu_{i+1,\cdot} \Leftrightarrow \nu_{i,k} \leq \nu_{i+1,k} \quad (k = 1, 2, \ldots, n),$$

$$|\nu_{i,\cdot}| = \nu_{i,1} + \nu_{i,2} + \ldots + \nu_{i,n}, X^{\nu_{i,\cdot}} = X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} \ldots X_n^{\nu_{i,n}} \quad i \in \mathcal{Z}, n \in \mathcal{N},$$

and $\mathcal{N}$, $\mathcal{Z}$, $\mathcal{R}$ are the sets of all natural, integer, and real numbers, respectively.

We say that the (n-quotient) geometrical net is an M-granule (n-quotient) tone system if $X^{M,\cdot} = 2$ (octave). The M-granule (n-quotient) tone system is scale (an M-granule n-quotient scale) if it is a chain, a sequence.

In this paper, we will consider the case of geometrical nets with $n = 3$ and $M = 12$. 12-granulation of octave is typical for European music. Both Slendro and Pelog can be derived from a theoretical 10-equally tempered master scale, i.e. music of the Pacific region uses 10-granulation within octave.

2 Unimodular (12, 7, 4)-matrices

Definition 1 We say that a matrix

$$A = \begin{pmatrix} \nu_{12,1} & \nu_{12,2} & \nu_{12,3} \\ \nu_{7,1} & \nu_{7,2} & \nu_{7,3} \\ \nu_{4,1} & \nu_{4,2} & \nu_{4,3} \end{pmatrix}$$

of nonnegative integers, is a (12, 7, 4)-*matrix*, if

$$0 \leq \nu_{4,1} \leq \nu_{7,1} \leq \nu_{12,1}, 0 \leq \nu_{4,2} \leq \nu_{7,2} \leq \nu_{12,2}, 0 \leq \nu_{4,3} \leq \nu_{7,3} \leq \nu_{12,3}$$

and

$$\nu_{i,1} + \nu_{i,2} + \nu_{i,3} = i, i = 4, 7, 12.$$

In other words, we say that a matrix $(\nu_{i,j})^{1,2,3}_{12,7,4} \in \mathcal{N}^3 \times \mathcal{N}^3$ is a (12, 7, 4)-*matrix*, cf. [8], if $0 \leq \nu_{4,\cdot} \leq \nu_{7,\cdot} \leq \nu_{12,\cdot}$ and $|\nu_{i,\cdot}| = i, i = 12, 7, 4$.

Definition 2 If to a given (12, 7, 4)-matrix A there exists an M-granule n-quotient scale S (tone system $\mathcal{S}$), then we say that the scale S (tone system $\mathcal{S}$) is *generated* by A.

We will find appropriate (12, 7, 4)-matrices and then construct and consider the needed generated 12-granule 3-quotient systems.

Theorem 1 *Let* $A = (\nu_{i,j})^{j=1,2,3}_{i=12,7,4} \in \mathcal{N}^3 \times \mathcal{N}^3$ *with* $\det A \neq 0$.
Then there exists a unique $X \in \mathcal{Q}^3$*, such that*

$$X^{\nu_{12,\cdot}} = 2/1, X^{\nu_{7,\cdot}} = 3/2, X^{\nu_{4,\cdot}} = 5/4, \tag{1}$$

and the following statements are equivalent:

(i) $X \in \mathcal{Q}^3$*;*

(ii) $\det A = 1$,

where $\mathcal{Q}$ *denotes the set of all rational numbers. The values are as follows:*
$X_1 = \sqrt[\det A]{2^{D_{2,1}} 3^{D_{3,1}} 5^{D_{5,1}}}, X_2 = \sqrt[\det A]{2^{D_{2,2}} 3^{D_{3,2}} 5^{D_{5,2}}}, X_3 = \sqrt[\det A]{2^{D_{2,3}} 3^{D_{3,3}} 5^{D_{5,3}}}$,
where

$$D_{2,1} = \begin{vmatrix} 1 & \nu_{12,2} & \nu_{12,3} \\ -1 & \nu_{7,2} & \nu_{7,3} \\ -2 & \nu_{4,2} & \nu_{4,3} \end{vmatrix}, D_{2,2} = \begin{vmatrix} \nu_{12,1} & 1 & \nu_{12,3} \\ \nu_{7,1} & -1 & \nu_{7,3} \\ \nu_{4,1} & -2 & \nu_{4,3} \end{vmatrix}, D_{2,3} = \begin{vmatrix} \nu_{12,1} & \nu_{12,2} & 1 \\ \nu_{7,1} & \nu_{7,2} & -1 \\ \nu_{4,1} & \nu_{4,2} & -2 \end{vmatrix},$$

$$D_{3,1} = \begin{vmatrix} 0 & \nu_{12,2} & \nu_{12,3} \\ 1 & \nu_{7,2} & \nu_{7,3} \\ 0 & \nu_{4,2} & \nu_{4,3} \end{vmatrix}, D_{3,2} = \begin{vmatrix} \nu_{12,1} & 0 & \nu_{12,3} \\ \nu_{7,1} & 1 & \nu_{7,3} \\ \nu_{4,1} & 0 & \nu_{4,3} \end{vmatrix}, D_{3,3} = \begin{vmatrix} \nu_{12,1} & \nu_{12,2} & 0 \\ \nu_{7,1} & \nu_{7,2} & 1 \\ \nu_{4,1} & \nu_{4,2} & 0 \end{vmatrix},$$

$$D_{5,1} = \begin{vmatrix} 0 & \nu_{12,2} & \nu_{12,3} \\ 0 & \nu_{7,2} & \nu_{7,3} \\ 1 & \nu_{4,2} & \nu_{4,3} \end{vmatrix}, D_{5,2} = \begin{vmatrix} \nu_{12,1} & 0 & \nu_{12,3} \\ \nu_{7,1} & 0 & \nu_{7,3} \\ \nu_{4,1} & 1 & \nu_{4,3} \end{vmatrix}, D_{5,3} = \begin{vmatrix} \nu_{12,1} & \nu_{12,2} & 0 \\ \nu_{7,1} & \nu_{7,2} & 0 \\ \nu_{4,1} & \nu_{4,2} & 1 \end{vmatrix}.$$

Proof. $(i) \Rightarrow (ii)$ If $X_1, X_2, X_3 \in \mathcal{Q}$, then there exist $p \in \mathcal{P}$ and $E_{2,X_1}, \ldots, E_{p,X_3} \in \mathcal{Z}$ such that

$$\begin{array}{l} X_1 = 2^{E_{2,X_1}} 3^{E_{3,X_1}} \ldots p^{E_{p,X_1}}, \\ X_2 = 2^{E_{2,X_2}} 3^{E_{3,X_2}} \ldots p^{E_{p,X_2}}, \\ X_3 = 2^{E_{2,X_3}} 3^{E_{3,X_3}} \ldots p^{E_{p,X_3}}. \end{array} \tag{2}$$

Combining (1) and (2),

$$\begin{pmatrix} E_{2,X_1} & E_{2,X_2} & E_{2,X_3} \\ E_{3,X_1} & E_{3,X_2} & E_{3,X_3} \\ E_{5,X_1} & E_{5,X_2} & E_{5,X_3} \\ E_{7,X_1} & E_{7,X_2} & E_{7,X_3} \\ & \ldots & \\ E_{p,X_1} & E_{p,X_2} & E_{p,X_3} \end{pmatrix} \begin{pmatrix} \nu_{12,1} & \nu_{7,1} & \nu_{4,1} \\ \nu_{12,2} & \nu_{7,2} & \nu_{4,2} \\ \nu_{12,3} & \nu_{7,3} & \nu_{4,3} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ & \ldots & \\ 0 & 0 & 0 \end{pmatrix}.$$

$(ii) \Rightarrow (i)$ Since $\det A = 1$,

$$\begin{pmatrix} E_{2,X_1} & E_{2,X_2} & E_{2,X_3} \\ E_{3,X_1} & E_{3,X_2} & E_{3,X_3} \\ E_{5,X_1} & E_{5,X_2} & E_{5,X_3} \\ E_{7x} & E_{7,X_2} & E_{7,X_3} \\ & \ldots & \\ E_{p,X_1} & E_{p,X_2} & E_{p,X_3} \end{pmatrix} = \begin{pmatrix} D_{2,X_1} & D_{2,X_2} & D_{2,X_3} \\ D_{3,X_1} & D_{3,X_2} & D_{3,X_3} \\ D_{5,X_1} & D_{5,X_2} & D_{5,X_3} \\ 0 & 0 & 0 \\ & \ldots & \\ 0 & 0 & 0 \end{pmatrix}.$$

$\square$

Theorem 2 *No 12-granule 1- or 2- rational interval system is generated by any (12, 7, 4)-matrix.*

Proof. The analysis of all (12, 7, 4)-matrices with $\det A = 1$ contains no case $D_{2,X_1} = D_{3,X_1} = D_{5,X_1} = 0$. So, $D^2_{2,X_1} + D^2_{3,X_1} + D^2_{5,X_1} > 0$. Analogously, $D^2_{2,X_2} + D^2_{3,X_2} + D^2_{5,X_2} > 0, D^2_{2,X_3} + D^2_{3,X_3} + D^2_{5,X_3} > 0$. The assertion for 1-interval system is trivial. $\square$

Corollary 3 *Neither the 12-tone Equal Temperament nor Pythagorean Tuning are generated by any (12, 7, 4)-matrix.*

Theorem 4 *Let A be a (12, 7, 4)-matrix. Then*

$$D^2_{2,X_1}+D^2_{2,X_2}+D^2_{2,X_3} > 0, D^2_{3,X_1}+D^2_{3,X_2}+D^2_{3,X_3} > 0, D^2_{5,X_1}+D^2_{5,X_2}+D^2_{5,X_3} > 0,$$

where $D_{2,X_1}, D_{2,X_2}, D_{2,X_3}, D_{3,X_1}, D_{3,X_2}, D_{3,X_3}, D_{5,X_1}, D_{5,X_2}, D_{5,X_3}$ *are as in Theorem 1.*

Proof. At first, note that a (12, 7, 4)-matrix exists, e.g. the matrix A_* in Theorem 8. If $D^2_{5,X_1} + D^2_{5,X_2} + D^2_{5,X_3} = 0$, then there exist no $\nu_{4,1}, \nu_{4,2}, \nu_{4,3}$ in A such that $X_1^{\nu_{4,1}} X_2^{\nu_{4,2}} X_3^{\nu_{4,3}} = 5/4$. A contradiction. Similarly for the numbers 2 and 3. $\square$

Theorem 5 *Let A be an unimodular (12, 7, 4)-matrix. Then $X_1 \neq X_2, X_2 \neq X_3, X_3 \neq X_1$.*

Proof. If $1 < X_1 = X_2 = X_3$, then $X_1 \notin \mathcal{Q}$. A contradiction.

Suppose $X_1 = X_2 \neq Z$ (the cases $X_2 = X_3 \neq X_1$, $X_1 = X_3 \neq X_2$ are symmetric). By Definition 1 and Definition 2,

$$X_1^{4-\nu_{4,3}} X_3^{\nu_{4,3}} = 5/4, X_1^{7-\nu_{7,3}} X_3^{\nu_{7,3}} = 3/2, X_1^{12-\nu_{12,3}} X_3^{\nu_{12,3}} = 2/1. \tag{3}$$

By Theorem 1, $X_1 = 2^\alpha 3^\beta 5^\gamma, X_3 = 2^\delta 3^\epsilon 5^\theta$, for some $\alpha, \beta, \gamma, \delta, \epsilon, \theta \in \mathcal{Z}$. Then (3) implies

$$\alpha(12-\nu_{12,3})+\nu_{12,3}\delta = 1, \beta(12-\nu_{12,3})+\nu_{12,3}\epsilon = 0, \gamma(12-\nu_{12,3})+\nu_{12,3}\theta = 0, \tag{4}$$

$$\alpha(7-\nu_{7,3}) + \nu_{7,3}\delta = -1, \beta(7-\nu_{7,3}) + \nu_{7,3}\epsilon = 1, \gamma(7-\nu_{7,3}) + \nu_{7,3}\theta = 0, \tag{5}$$

$$\alpha(4-\nu_{4,3}) + \nu_{4,3}\delta = -2, \beta(4-\nu_{4,3}) + \nu_{4,3}\epsilon = 0, \gamma(4-\nu_{4,3}) + \nu_{4,3}\theta = 1. \tag{6}$$

If $\epsilon = \beta$, then (4) implies $\beta = 1/7 \notin \mathcal{Z}$. If $\gamma = \theta$, then (5) implies $\gamma = 1/4 \notin \mathcal{Z}$. So, $\epsilon \neq \beta, \theta \neq \gamma$. Then (4), and (5) imply

$$\nu_{12,3} = \frac{-12\beta}{\epsilon - \beta} = \frac{-12\gamma}{\theta - \gamma}, \nu_{4,3} = \frac{-4\beta}{\epsilon - \beta} = \frac{1 - 4\gamma}{\theta - \gamma}. \tag{7}$$

If $\beta \neq 0$, then (6) implies

$$\frac{-12\beta}{-4\beta} = \frac{-12\gamma}{1 - 4\gamma},$$

which implies $0 = 1$. If $\beta = 0$ then (6) implies $0 = \gamma = 1/4$. A contradiction. $\square$

Corollary 6 *If S ($\mathcal{S}$) is a 12-granule 3-quotient (2/1, 3/2, 5/4)-scale (system) with $X_1, X_2, X_3 \in \mathcal{Q}$, then we can redenote (order) X_1, X_2, X_3, such that $1 < X_1 < X_2 < X_3 < 10/9$.*

3 TDS geometrical nets

Now, we restrict the class of all 12-granule 3-quotient geometrical nets generated by (12, 7, 4)-matrices to TDS geometrical nets.

Definition 3 We say that a 12-granule 3-quotient geometrical net $\langle \Gamma_i \rangle$ generated by a (12, 7, 4)-matrix A is a *TDS* geometrical net if

$$\nu_{2,\cdot} = 2\nu_{7,\cdot} - \nu_{12,\cdot}, \nu_{5,\cdot} = \nu_{12,\cdot} - \nu_{7,\cdot}, \nu_{9,\cdot} = \nu_{12,\cdot} - \nu_{7,\cdot} + \nu_{4,\cdot}, \nu_{11,\cdot} = \nu_{7,\cdot} + \nu_{4,\cdot},$$

and for every $i \in \mathcal{Z}$, there exists $p \in \mathcal{N}, 0 \leq p < 12$, and $q \in \mathcal{N}$, such that $\nu_{i,\cdot} = q\nu_{12,\cdot} + \nu_{p,\cdot}$.

Note that the members $\Gamma_i, i = 1, 3, 6, 8, 10$, mentioned in Definition 3, are determined ambiguously.

Theorem 7 *According to the symmetry, all generators $X \in \mathcal{Q}^3$ for TDS geometrical nets*

$$\langle \Gamma_i \rangle = \langle X^{\nu_{i,\cdot}}; |\nu_{i,\cdot}| = i, 0 \leq \nu_{0,\cdot} \leq \nu_{1,\cdot} \leq \ldots \leq \nu_{i,\cdot} \leq \ldots \rangle_{\nu_{i,\cdot} \in \mathcal{N}^3}$$

with the subsequences

$$\langle \Gamma_{12l} \rangle = \left\langle 2^l \right\rangle, \langle \Gamma_{12l+7} \rangle = \left\langle 3 \cdot 2^{l-1} \right\rangle, \langle \Gamma_{12l+4} \rangle = \left\langle 5 \cdot 2^{l-2} \right\rangle$$

are the following:

$$(25/24, 135/128, 16/15), (256/243, 135/128, 16/15), (25/24, 16/15, 27/25).$$

Proof. The analysis of the Diophantine equation

$$\det[(\nu_{i,j})_{i=12,7,4}^{j=1,2,3}] = 1, 0 \leq \nu_{4,\cdot} \leq \nu_{7,\cdot} \leq \nu_{12,\cdot}, |\nu_{i,\cdot}| = i$$

in $\mathcal{N}^3 \times \mathcal{N}^3$ with the additional (not restricting the solution) condition

$$2\nu_{7,\cdot} - \nu_{12,\cdot} \geq 0$$

yields the following matrices (excluding symmetries, i.e. permutations of columns):

$$A_1 = \begin{pmatrix} 2 & 7 & 3 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 2 & 5 & 5 \\ 1 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 5 & 4 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 2 & 9 \\ 1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix}, A_6 = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} 1 & 5 & 6 \\ 1 & 3 & 3 \\ 1 & 2 & 1 \end{pmatrix}, A_8 = \begin{pmatrix} 2 & 3 & 7 \\ 1 & 2 & 4 \\ 1 & 0 & 3 \end{pmatrix}.$$

Apply Theorem 1 and find all sequences by the algorithm in Definition 3. Excluding all such sequences $\langle \Gamma_i \rangle$ which do not satisfy the condition $\nu_{0,\cdot} \leq \nu_{1,\cdot} \leq \ldots \leq \nu_{i,\cdot} \leq \ldots$, we obtain the following three matrices: A_1, A_2, A_3.

In Table 1, Table 2, and Table 3 there are all TDS-geometrical nets $\langle \Gamma_i \rangle$ (in the sixths column, there is a musical denotation) corresponding to the matrices A_1, A_2, and A_3. □

In the connection with the previous theorem we mention here that the analysis of all (12, 7, 4)-matrices A with $\det A = 1$ yields the following surprising statement.

Theorem 8 *According to the symmetry,*

$$A_3 = \begin{pmatrix} 5 & 4 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

is the unique unimodular matrix which is the unique solution of the Diophantine equation $\det[(\nu_{i,j})_{i=12,7,4}^{j=1,2,3}] = 1, 0 < \nu_{4,\cdot} < \nu_{7,\cdot} < \nu_{12,\cdot}, |\nu_{i,\cdot}| = i$.

Corollary 9 *By Corollary 6, let* $1 < X < X_2 < X_3 < 10/9$*. By Theorem 1, for* A_3 *we have:*

$$\begin{pmatrix} D_{2,X_1} & D_{2,X_2} & D_{2,X_3} \\ D_{3,X_1} & D_{3,X_2} & D_{3,X_3} \\ D_{5,X_1} & D_{5,X_2} & D_{5,X_3} \end{pmatrix} = \begin{pmatrix} -3 & 4 & 0 \\ -1 & -1 & 3 \\ 2 & -1 & -2 \end{pmatrix}.$$

4 Construction of generated tone systems

In this section we will generate 12-granule 3-quotient tone systems.

The values $X, X_2, X_3 \in \mathcal{R}$ in the following theorem not be necessary rational.

Theorem 10 *Let A be a (12, 7, 4)-matrix and corresponding* $X, X_2, X_3 \in \mathcal{R}$ *as in Theorem 1.*

Put

$\nu^*_{2,1} = 2\nu_{7,1} - \nu_{12,1}, \nu^*_{2,2} = 2\nu_{7,2} - \nu_{12,2}, \nu^*_{2,3} = 2\nu_{7,3} - \nu_{12,3}$,
$\nu^*_{5,1} = \nu_{12,1} - \nu_{7,1}, \nu^*_{5,2} = \nu_{12,2} - \nu_{7,2}, \nu^*_{5,3} = \nu_{12,3} - \nu_{7,3}$,
$\nu^*_{9,1} = \nu_{12,1} - \nu_{7,1} + \nu_{4,1}, \nu^*_{9,2} = \nu_{12,2} - \nu_{7,2} + \nu_{4,2}, \nu^*_{9,3} = \nu_{12,3} - \nu_{7,3} + \nu_{4,3}$,
$\nu^*_{11,1} = \nu_{7,1} + \nu_{4,1}, \nu^*_{11,2} = \nu_{7,2} + \nu_{4,2}, \nu^*_{11,3} = \nu_{7,3} + \nu_{4,3}$.

...	...	...	...	...
$X_1^0X_2^0X_3^0$	$2^03^05^0$	1/1	1.0	C
$X_1^1X_2^0X_3^0$	$2^{-3}3^{-1}5^2$	25/24	1.041666666	$C_\sharp$
$X_1^0X_2^0X_3^1$	$2^03^35^{-2}$	27/25	1.08	$D_\flat$
$X_1^1X_2^0X_3^1$	$2^{-3}3^25^0$	9/8	1.125	D
$X_1^2X_2^0X_3^1$	$2^{-6}3^15^2$	75/64	1.171875	$D_\sharp$
$X_1^1X_2^1X_3^1$	$2^13^15^{-1}$	6/5	1.2	$E_\flat$
$X_1^2X_2^1X_3^1$	$2^{-2}3^05^1$	5/4	1.25	E
$X_1^2X_2^2X_3^1$	$2^23^{-1}5^0$	4/3	1.333333333	F
$X_1^3X_2^2X_3^1$	$2^{-1}3^{-2}5^2$	25/18	1.388888888	$F_\sharp$
$X_1^2X_2^2X_3^2$	$2^23^25^{-2}$	36/25	1.44	$G_\flat$
$X_1^3X_2^2X_3^2$	$2^{-1}3^15^0$	3/2	1.5	G
$X_1^4X_2^2X_3^2$	$2^{-4}3^05^2$	25/16	1.5625	$G_\sharp$
$X_1^3X_2^3X_3^2$	$2^33^05^{-1}$	8/5	1.6	$A_\flat$
$X_1^4X_2^3X_3^2$	$2^03^{-1}5^1$	5/3	1.666666666	A
$X_1^5X_2^3X_3^2$	$2^{-3}3^{-2}5^3$	125/72	1.736111111	$A_\sharp$
$X_1^4X_2^3X_3^3$	$2^03^25^{-1}$	9/5	1.8	$B_\flat$
$X_1^5X_2^3X_3^3$	$2^{-3}3^15^1$	15/8	1.875	B
$X_1^5X_2^4X_3^3$	$2^13^05^0$	2/1	2.0	C'
...	...	...	...	...

Table 1: Class G, semitones $(X_1, X_2, X_3) = (25/24, 16/15, 27/25)$

...	...	...	...	...
$X_1^0X_2^0X_4^0$	$2^03^05^0$	1/1	1.0	C
$X_1^0X_2^0X_4^1$	$2^{-7}3^351$	135/128	1.0546875	$C_\sharp$
$X_1^0X_2^1X_4^0$	$2^43^{-1}5^{-1}$	16/15	1.066666666	$D_\flat$
$X_1^0X_2^1X_4^1$	$2^{-3}3^25^0$	9/8	1.125	D
$X_1^1X_2^1X_4^1$	$2^{-6}3^15^2$	75/64	1.171875	$D_\sharp$
$X_1^0X_2^2X_4^1$	$2^13^15^{-1}$	6/5	1.2	$E_\flat$
$X_1^1X_2^2X_4^1$	$2^{-2}3^05^1$	5/4	1.25	E
$X_1^1X_2^3X_4^1$	$2^23^{-1}5^0$	4/3	1.333333333	F
$X_1^1X_2^3X_4^2$	$2^{-5}3^25^1$	45/32	1.40625	$F_\sharp$
$X_1^1X_2^4X_4^1$	$2^63^{-2}5^{-1}$	64/45	1.422222222	$G_\flat$
$X_1^1X_2^4X_4^2$	$2^{-1}3^15^0$	3/2	1.5	G
$X_1^2X_2^4X_4^2$	$2^{-4}3^05^2$	25/16	1.5625	$G_\sharp$
$X_1^1X_2^5X_4^2$	$2^33^05^{-1}$	8/5	1.6	$A_\flat$
$X_1^2X_2^5X_4^2$	$2^03^{-1}5^1$	5/3	1.666666666	A
$X_1^2X_2^5X_4^3$	$2^{-7}3^25^2$	225/128	1.7578125	$A_\sharp$
$X_1^2X_2^6X_4^2$	$2^43^{-2}5^0$	16/9	1.777777777	$B_\flat$
$X_1^2X_2^6X_4^3$	$2^{-3}3^15^1$	15/8	1.875	B
$X_1^2X_2^7X_4^3$	$2^13^05^0$	2/1	2.0	C'
...	...	...	...	...

Table 2: Class R, semitones $(X_1, X_2, X_4) = (25/24, 16/15, 135/128)$

$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$X_5^0 X_2^0 X_4^0$	$2^0 3^0 5^0$	1/1	1.0	C
$X_5^0 X_2^0 X_4^1$	$2^{-7} 3^3 5^1$	135/128	1.0546875	$C_\sharp$
$X_5^0 X_2^1 X_4^0$	$2^4 3^{-1} 5^{-1}$	16/15	1.066666666	$D_\flat$
$X_5^0 X_2^1 X_4^1$	$2^{-3} 3^2 5^0$	9/8	1.125	D
$X_5^1 X_2^1 X_4^1$	$2^5 3^{-3} 5^0$	32/27	1.185185185	$D_\sharp$
$X_5^0 X_2^1 X_4^2$	$2^{-10} 3^5 5^1$	1215/1024	1.186523438	$E_\flat$
$X_5^1 X_2^1 X_4^2$	$2^{-2} 3^0 5^1$	5/4	1.25	E
$X_5^1 X_2^2 X_4^2$	$2^2 3^{-1} 5^0$	4/3	1.333333333	F
$X_5^1 X_2^2 X_4^3$	$2^{-5} 3^2 5^1$	45/32	1.40625	$F_\sharp$
$X_5^1 X_2^3 X_4^2$	$2^6 3^{-2} 5^{-1}$	64/45	1.422222222	$G_\flat$
$X_5^1 X_2^3 X_4^3$	$2^{-1} 3^1 5^0$	3/2	1.5	G
$X_5^2 X_2^3 X_4^3$	$2^7 3^{-4} 5^0$	128/81	1.580246914	$G_\sharp$
$X_5^1 X_2^3 X_4^4$	$2^{-8} 3^4 5^1$	405/256	1.58203125	$A_\flat$
$X_5^2 X_2^3 X_4^4$	$2^0 3^{-1} 5^1$	5/3	1.666666666	A
$X_5^2 X_2^3 X_4^5$	$2^{-7} 3^2 5^2$	225/128	1.7578125	$A_\sharp$
$X_5^2 X_2^4 X_4^4$	$2^4 3^{-2} 5^0$	16/9	1.777777777	$B_\flat$
$X_5^2 X_2^4 X_4^5$	$2^{-3} 3^1 5^1$	15/8	1.875	B
$X_5^2 X_2^5 X_4^5$	$2^1 3^0 5^0$	2/1	2.0	C'
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Table 3: Class P, semitones $(X_5, X_2, X_4) = (256/243, 16/15, 135/128)$

Put

$C = X_1^0 X_2^0 X_3^0$, $D = X_1^{\nu^*_{2,1}} X_2^{\nu^*_{2,2}} X_3^{\nu^*_{2,3}}$, $E = X_1^{\nu_{4,1}} X_2^{\nu_{4,2}} X_3^{\nu_{4,3}}$, $F = X_1^{\nu^*_{5,1}} X_2^{\nu^*_{5,2}} X_3^{\nu^*_{5,3}}$, $G = X_1^{\nu_{7,1}} X_2^{\nu_{7,2}} X_3^{\nu_{7,3}}$, $A = X_1^{\nu^*_{9,1}} X_2^{\nu^*_{9,2}} X_3^{\nu^*_{9,3}}$, $B = X_1^{\nu^*_{11,1}} X_2^{\nu^*_{11,2}} X_3^{\nu^*_{11,3}}$, $C' = X_1^{\nu_{12,1}} X_2^{\nu_{12,2}} X_3^{\nu_{12,3}}$, $D' = 2d$.

Then

(i) $C : E : G = G : B : D' = F : A : C' = 1 : 5/4 : 3/2$,

(ii) $\nu^*_{2,1} + \nu^*_{2,2} + \nu^*_{2,3} = 2, \nu^*_{5,1} + \nu^*_{5,2} + \nu^*_{5,3} = 5, m^*_{9,1} + \nu^*_{9,2} + \nu^*_{9,3} = 9, \nu^*_{11,1} + \nu^*_{11,2} + \nu^*_{11,3} = 11.$

Proof. (i) We have: $1 : (5/4) : (3/2) = C : E : G = 3/2 : ((3/2)(5/4)) : (3/2)^2 = G : B : D' = ((2)(2/3)) : ((2)(2/3)(5/4)) : 2 = F : A : C'$.

(ii) For 5, $(\nu_{12,1} - \nu_{7,1}) + (\nu_{12,2} - \nu_{7,2}) + (\nu_{12,3} - \nu_{7,3}) = (\nu_{12,1} + \nu_{12,2} + \nu_{12,3}) - (\nu_{7,1} + \nu_{7,2} + \nu_{7,3}) = 12 - 7 = 5$, analogously for 2, 9, 11. $\square$

Corollary 11 *If*

$$\begin{array}{llllllll} 0 = m_0 & \leq \nu^*_{2,1} & \leq \nu_{4,1} & \leq \nu^*_{5,1} & \leq \nu_{7,1} & \leq \nu^*_{9,1} & \leq \nu^*_{11,1} & \leq \nu_{12,1}, \\ 0 = n_0 & \leq \nu^*_{2,2} & \leq \nu_{4,2} & \leq \nu^*_{5,2} & \leq \nu_{7,2} & \leq \nu^*_{9,2} & \leq \nu^*_{11,2} & \leq \nu_{12,2}, \\ 0 = r_0 & \leq \nu^*_{2,3} & \leq \nu_{4,3} & \leq \nu^*_{5,3} & \leq \nu_{7,3} & \leq \nu^*_{9,3} & \leq \nu^*_{11,3} & \leq \nu_{12,3}, \end{array}$$

then numbers D, F, A, B can be taken as the 3rd, 6th, 10th, and 12th coordinates of the 12-granule scale vector and put

$\nu^*_{2,1} = \nu_{2,1}, \nu^*_{2,2} = \nu_{2,2}, \nu^*_{2,3} = \nu_{2,3}$,
$\nu^*_{5,1} = \nu_{5,1}, \nu^*_{5,2} = \nu_{5,2}, \nu^*_{5,3} = \nu_{5,3}$,
$\nu^*_{9,1} = \nu_{9,1}, \nu^*_{9,2} = \nu_{9,2}, \nu^*_{9,3} = \nu_{9,3}$,
$\nu^*_{11,1} = \nu_{11,1}, \nu^*_{11,2} = \nu_{11,2}, \nu^*_{11,3} = \nu_{11,3}$.

Theorem 12 *Let* $(X_1, X_2, X_3) = (25/24, 16/15, 27/25)$. *Let* $C = X_1^0 X_2^0 X_3^0$, $C_\sharp = X_1^1 X_2^0 X_3^0$, $D_\flat = X_1^0 X_2^0 X_3^1$, $D = X_1^1 X_2^0 X_3^1$, $D_\sharp = X_1^2 X_2^0 X_3^1$, $E_\flat = X_1^1 X_2^1 X_3^1$, $E = X_1^2 X_2^1 X_3^1$, $F = X_1^2 X_2^2 X_3^1$, $F_\sharp = X_1^2 X_2^2 X_3^2$, $G_\flat = X_1^3 X_2^2 X_3^1$, $G = X_1^3 X_2^2 X_3^2$, $G_\sharp = X_1^4 X_2^2 X_3^2$, $A_\flat = X_1^3 X_2^3 X_3^2$, $A = X_1^4 X_2^3 X_3^2$, $A_\sharp = X_1^5 X_2^3 X_3^2$, $B_\flat = X_1^4 X_2^3 X_3^3$, $B = X_1^5 X_2^3 X_3^3$, $C' = X_1^5 X_2^4 X_3^3$, $D' = 2d$.

Then the all 12-granule scales generated by A_3 *satisfying the condition* $C : E : G = G : B : D' = F : A : C' = 1 : 5/4 : 3/2$ *are the next:*

$$(C, i, D, j, E, F, k, G, l, A, m, B, C'),$$

where $i = C_\sharp, D_\flat; j = D_\sharp, E_\flat; k = F_\sharp, G_\flat; l = G_\sharp, A_\flat; m = A_\sharp, B_\flat$.

Analogously for matrices A_1 *(cf. Table 2) and* A_2 *(cf. Table pypri), respectively.*

Proof. Combine Corollary 6, Theorem 10, and Corollary 11. It is easy to verify, cf. Table 1, that $i = C_\sharp, D_\flat; j = D_\sharp, E_\flat; k = F_\sharp, G_\flat; l = G_\sharp, A_\flat; m = A_\sharp, B_\flat$, are the all possibilities how to complete $\{C, D, E, F, G, A, B, C'\}$ to the 12-granule scales. □

Corollary 13 *The tone system*

$$\mathcal{S}_\ni = \{C, C_\sharp, D_\flat, D, D_\sharp, E_\flat, E, F, F_\sharp, G_\flat, G, G_\sharp, A_\flat, A, A_\sharp, B_\flat, B, C'\},$$

cf. Table 1, is a 17-valued 12-granule 3-quotient (2/1, 3/2, 5/4)-system.

Observe that the structure of $\mathcal{S}_\ni$ is similar to the 17-valued Pythagorean Tuning (two values for "black keys" on the standard keyboard).

5 Comment to superparticular ratios

In this section we show that the found systems (and $\mathcal{S}_\ni$ in particular) is very near also to Just Intonation.

The only pairs of naturals $(N+1, N)$, for which $(N+1)$ and $N, N \in \mathcal{N}$, are divisible only by 2, 3, or 5, are

$$(2,1), (3,2), (4,3), (5,4), (6,5), (9,8), (10,9), (16,15), (25,24), (81,80).$$

The following *superparticular ratios*, cf. [15],

$$2/1, 3/2, 4/3, 5/4, 6/5, 9/8, 10/9, 16/15, 25/24, 81/80$$

	(X_1, X_2, X_3)	(X_1, X_4, X_2)	(X_5, X_4, X_2)
2/1	$X_1^5 X_2^4 X_3^3$	$X_1^2 X_4^3 X_2^7$	$X_5^2 X_4^5 X_2^5$
3/2	$X_1^3 X_2^2 X_3^2$	$X_1 X_4^2 X_2^4$	$X_5 X_4^3 X_2^3$
4/3	$X_1^2 X_2^2 X_3$	$X_1 X_4 X_2^3$	$X_5 X_4 2 X_2^2$
5/4	$X_1^2 X_2 X_3$	$X_1 X_4 X_2^2$	$X_5 X_4^2 X_2$
6/5	$X_1 X_2 X_3$	$X_4 X_2^2$	$X_4 X_2^2$
9/8	$X_1 X_3$	$X_4 X_2$	$X_4 X_2$
10/9	$X_1 X_2$	$X_1 X_2$	$X_5 X_4$
16/15	X_2	X_2	X_2
25/24	X_1	X_1	$X_5 X_4 X_2^{-1}$
81/80	$X_1^{-1} X_3$	$X_1^{-1} X_4$	$X_5^{-1} X_2$

Table 4: Semitones for superparticular ratios: $X_1 = 25/24, X_2 = 16/15, X_3 = 27/25, X_4 = 135/128, X_5 = 256/243$

account for common music intervals (they denominate the relative acoustic frequency or, inversely, the length of the pipe or the string) and correspond to *octave*, *perfect fifth*, *perfect fourth*, *major third*, *minor third*, *major whole tone*, *minor whole tone*, *diatonic semitone*, *chromatic semitone*, and *comma of Dydimus*, respectively.

The proof of the following theorem is easy.

Theorem 14 *See Table 4.*

6 Classification of diatonic scales

We split the set of all 12-granule 3-quotient tone systems which contain C major scale into three classes. Denote them P, G, R, respectively. We show that they can be characterized as follows: the class G contains Gypsy scales, the class P – Pythagorean heptatonic, and the class R – Redfield scale.

By a *diatonic scale* we mean usually a 7-tone scale within the octave in which the neighbouring intervals are not smaller than a semitone and not greater than three semitones (= the *hiat*).

There are many different semitones in music. Each of them has its own good reason for existence (depending on the temperature of the scale). Some examples of semitones: *Pythagorean minor semitone* (256/243), *Pythagorean major semitone* (2187/2048), *Diatonic semitone* (16/15), *Chromatic semitone* (25/24), *Praetorius minor semitone* ($\sqrt[4]{78125}/16$), *Praetorius major semitone* ($8/\sqrt[4]{3125}$), *Co-chromatic semitone* (27/25), *Co-diatonic semitone* (135/128), *Equal-tempered semitone* ($\sqrt[12]{2}$), etc.

The appearance of other intervals between neighbouring tones (the whole tones and/or hiats) in diatonic scales depends on the used semitones.

Various scales (diatonic or nondiatonic) use various numbers of different semitones. Equal temperament uses one semitone. Pythagorean Tuning is constructed by two semitones, analogously Praetorius Tuning (1/4-comma meantone). What about diatonic scales in general? It is known that the typical diatonic scale, the major scale, is constructed e.g. by the semitones: 16/15, 25/24, and 27/25. Thus, not greater that three semitones are needed for constructing of diatonic scales. But there are more than one possibilities as we have seen in the previous sections (the triples $(16/15, 25/24, 135/128)$ and $(256/243, 16/15, 135/128)$ of semitones can also serve for constructing of the major diatonic scale).

From the dimensional point of view in the Euler musical space, Equal Temperament can be imaged in the line [the reper: the octave], Pythagorean Tuning (Praetorius Tuning) in the plane [the repers: the octave and perfect fifth (the octave and major third)], and the diatonic major scale [the repers: the octave, perfect fifth, major thirds] is a set of points in the 3-dimensional space. Note also that Just Intonation with the natural seventh, moreover, needs the fourth dimension, but it is not a diatonic tone system.

Although every musician understands what is a diatonic scale, we have find no mathematical definition in the literature. Certainly, the reason is the "big ambiguity" of this notion in music. We bring a definition of the notion of diatonic scale from the mathematical point of view.

We fix the structure of the 7-valued major scale (intervals between tones: 9/8, 10/9, 16/15, 9/8, 10/9, 9/8, 16/15) and enlarge it to 12-granule 3-quotient geometrical nets, i.e. the elements of the resulting tone system will be of the form:

$$\Gamma_i = X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} X_3^{\nu_{i,3}},$$

where $X_1, X_2, X_3 \in (2^{1/24}; 2^{3/24})$,

$$\Gamma_0 = 1, \Gamma_{12} = 2,$$

and $i, \nu_{i,1}, \nu_{i,2}, \nu_{i,3}$ are nonegative integers such that

$$\nu_{i,1} + \nu_{i,2} + \nu_{i,3} = i$$

and

$$0 \leq \nu_{1,j} \leq \nu_{2,j} \cdots \leq \nu_{n,j} \cdots,$$

$j = 1, 2, 3$, and $i \in \mathcal{N}$.

Further, we suppose the *octave equivalency*, i.e.

$$(\Gamma_{12i+0}, \Gamma_{12i+1}, \ldots, \Gamma_{12i+11}) = 2^i(\Gamma_0, \Gamma_1, \ldots, \Gamma_{11})$$

where i is a natural number.

The idea how to define the diatonic scales consists of (i) choosing a 7 valued variation from these nets according to octave equivalency, and (ii) possible apply a homomorphism of geometrical nets.

(i) According to octave equivalency, consider a variation

$$\mathcal{D} = (\Gamma_{i_1}, \Gamma_{i_2}, \Gamma_{i_3}, \Gamma_{i_4}, \Gamma_{i_5}, \Gamma_{i_6}, \Gamma_{i_7}, \Gamma_{i_8})$$

from the set

$$\{\Gamma_i; i = 0, 1, \ldots, 11\}$$

such that

$$1 \leq i_{n+1} - i_n \leq 3, n = 1, 2, 3, 4, 5, 6, 7.$$

(ii) Let $S_i = X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} X_3^{\nu_{i,3}}, Q_i = Y_1^{\mu_{i,1}} Y_2^{\mu_{i,2}} Y_3^{\mu_{i,3}}$ be two geometrical nets such that $S_0 = 1, S_{12} = 2, Q_0 = 1, Q_{12} = 2$, and $X_1, X_2, X_3, Y_1, Y_2, Y_3 \in (2^{1/24}; 2^{3/24})$. A map

$$\theta : (S_i) \to (Q_i)$$

is a *homomorphism of geometrical nets* (S_i) and (Q_i) if for every i nonnegative integer number, $S_i = X^{\nu_{i,1}} X_2^{\nu_{i,2}} X_3^{\nu_{i,3}} \Rightarrow Q_i = Y_1^{\nu_{i,1}} Y_2^{\nu_{i,2}} Y_3^{\nu_{i,3}}$.

By a *diatonic scale* we understand the variation $\mathcal{D}$ according to homomorphisms of geometrical nets.

The Table 1, Table 2, and Table 3 show the result of enlargement of the diatonic major scale $(C, D, E, F, G, A, B, C')$ to 12-granule 3-quotient scales. There are 96 12-granule 3-quotient scales such they are geometrical nets. From these 12-granule scales we choose diatonic scales (not considering homomorphism).

Denote the classes od diatonic scales given by Table 2, Table 1, and Table 3 as R, G, P corresponding triplets $(X_1, X_2, X_3), (X_1, X_2, X_4), (X_5, X_2, X_4)$, respectively.

Theorem 15 *The class R contains the Redfield diatonic scale.*

Proof. The Redfield diatonic scale is defined by the sequence of intervals between the neighbour tones: (10/9, 9/8, 16/15, 9/8, 10/9, 9/8, 16/15). We see, Table 2, that the sequence

$$(E_\flat, F, G, A_\flat, B_\flat, C', D', E'_\flat)$$

satisfies the requirement, where $D' = 2D, E_\flat = 2E'_\flat$. □

Theorem 16 *The class G contains the Gypsy major and minor scales.*

Proof.

(a) The Gypsy major scale is defined by the sequence of intervals between the neighbour tones: (16/15, 9/8 · 25/24, 16/15, 9/8, 16/15, 9/8 · 25/24, 16/15). We see, Table 1, that the sequence

$$(C, D_\flat, E, F, G, A_\flat, B, C')$$

satisfies the requirement.

(b) The Gypsy minor scale is defined by the sequence of intervals between the neighbour tones: (9/8, 16/15, 9/8 · 25/24, 16/15, 16/15, 9/8 · 25/24, 16/15). We see, Table 1, that the sequence

$$(A, B, C', D'_\sharp, E', F', G'_\sharp, A')$$

satisfies the requirement, where $D'_\sharp = D_\sharp, E' = 2E, F' = 2F$. □

Theorem 17 *The class P contains the Pythagorean heptatonic.*

Proof. The Pythagorean heptatonic is defined by the sequence of intervals between the neighbour tones: (9/8, 9/8, 256/243, 9/8, 9/8, 9/8, 256/243). We see, Table 3, that the sequence

$$(D_\sharp, F, G, G_\sharp, B_\flat, C', D', D'_\sharp)$$

satisfies the requirement, where $D' = 2D, D'_\sharp = 2D_\sharp$. □

The following three theorems can be verified directly.

Theorem 18 *The class G contains no Redfield scale and no Pythagorean heptatonic.*

Theorem 19 *The class R contains no Gypsy scale and no Pythagorean heptatonic.*

Theorem 20 *The class P contains no Redfield scale and no Gypsy scale.*

The other diatonic scales we obtain from classes G, P, R via homomorphism (and specially, isomorphism). We do not describe them in this paper and note now only some important special cases.

If $X_1 = X_2 = X_3 = X_4 = X_5 = \sqrt[12]{2}$, then Table 1, Table 2, and Table 3 define Equal Temperament. Another interesting simplification of the general case via homomorphism we obtain in the following theorem which can be verified directly.

Theorem 21 *If*
(a) $X_2 = X_3 = a, X_1 = b$ *(see Table 1), or*
(b) $X_1 = X_2 = a, X_4 = b$ *(see Table 2), or*
(c) $X_5 = X_2 = a, X_4 = b$ *(see Table 3),*
and $a = 256/243, b = 2187/2048$ *(or* $a = 8/\sqrt[4]{3125}, b = \sqrt[4]{78125}/16$*), then Table 1, Table 2, and Table 3 contain Pythagorean (or Praetorius) Tuning.*

Corollary 22 *Pythagorean Tuning (reduced to a 12-valued one) and Praetorius Tuning (1/4-comma meantone) are isomorphic, for an other approach, cf. [7].*

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