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MOMENT PROBLEM FOR MAJORATED OPERATORS II

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Abstract. We present a moment problem in the context of majorated operators on the space of continuous functions. A generalization of the Hausdorff moment problem theorem is formulated and proved and then applied to a representation of majorated linear mappings.

1. Introduction

It is well known that the following theorem is true.

Riesz representation theorem. *Every continuous linear functional L on the set of continuous functions f defined on $[0,1]$ has the form*

$$Lf = \int_0^1 f(s)dg(s) \tag{R}$$

with a function g of bounded variation on $[0,1]$.

This theorem has many extensions and generalizations with various proofs. One of the possible proofs is based on the moment problem theorem. It can be shown that [Wid 7] the problem of determining the general continuous linear functionals on the set of continuous functions is equivalent to that of determining the set of all moment sequences. It is our purpose to extend this result for majorated linear operators from continuous functions to Banach spaces ([DDD 2],[DH 4]).

2. A Helly theorem in Banach spaces

Recall that the function $g(t) : [a,b] \to Y$, Y being a normed space, is said to have bounded variation $|g|$ if $sup \sum_j ||g(s_j) - g(s_{j-1})|| < \infty$ where all possible partitions of $[a,b]$ are considered.

We will say that g has bounded semi-variation if the set

$$V_g = \{\sum_{j=1}^{k}(g(t_j) - g(t_{j-1}))\alpha_j \mid k \in N, a = t_0 < t_1 < \ldots < t_k = b, |\alpha_j| \leq 1\}$$

is a bounded set in X.

We will say that g has weakly compact semi-variation if g has bounded semi-variation and if the set V_g is included in a weakly compact set W of X.

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It is clear that, when g has weakly compact semi-variation, for any continuous function $f : [a, b] \to R$, we have $\int f \, dg \in \|f\| W$ if $V_g \subset W$.

If $V_g \subset W$, then $g([a, b]) \subset W + g(a)$.

A sequence $(g_n)_{n\in N}$ is uniformly of bounded (resp. weakly compact) semi-variation if there exists a bounded (resp. weakly compact) set containing V_{g_n} for any n.

In the following we shall need this form of a Helly theorem [DDD 2]

Helly theorem. *Let X be a Banach space, D a dense subset of $[a, b]$ and $(g_n)_{n\in N}$ a sequence of functions from $[a, b]$ into X such that*

a) $g_n(a) = 0$,

b) $(g_n)_{n\in N}$ is uniformly of weakly compact semi-variation.

Then there exists a subsequence of the sequence $(g_n)_{n\in N}$ weakly converging on D to a function $g : [a, b] \to X$ of weakly compact semi-variation.

Now we shall prove a version of Helly theorem for functions of bounded variation.

Helly theorem bis. *Let X be a Banach space, D a dense subset of $[a, b]$ and $(g_n)_{n\in N}$ a sequence of functions from $[a, b]$ into X such that*

a) $g_n(a) = 0$,

b) $(g_n)_{n\in N}$ is uniformly of bounded variation.

Then there exists a subsequence of the sequence $(g_n)_{n\in N}$ weakly converging on D to a function $g : [a, b] \to X$ of bounded variation.

To prove this theorem we shall require some lemmas.

Lemma 1. *Every function of bounded variation is a function of weakly compact semi-variation.*

Proof. It is known that if vector function g with bounded variation is also continuous from the left in the interior of interval I then the variation of vector measure corresponding to g coincides with the positive measure corresponding to the variation of g (cf. e.g. DI [3]). Now it is known that every vector measure on Borelian sets has relatively weakly compact range [BDS 1].

To prove the next lemma we make of use the following proposition.

Proposition. *Let $\{m_i\}$ be a sequence of vector-valued measures of uniformly bounded variation on Borelian sets $B[a, b]$ into a Banach space X. Then the set $\{m_i(E), E \in B[a, b], i = 1, 2, \dots\}$ is contained in some weakly compact subset of X, i. e. there exists a weakly compact subset W of X such that*

$$\{m_i(E), E \in \ B[a, b], i = 1, 2, \dots\} \subset W$$

i.e., the range of vector-valued measures m_i, $i = 1, 2, \dots$ is a relatively weakly compact subset of X.

Proof. The set $\{x'm_i, x' \in X', \|x'\| \leqq 1, i = 1, 2, \dots\}$ of scalar measures in the Banach space of all scalar measures on B with variation as norm, $ca(B)$, is

(1) bounded and

(2) if $\{E_n\}$ is a sequence in $B[a, b]$ which decreases to the empty set, then

$$\lim_{n\to\infty} x'm_i(E_n) = 0$$

uniformly for $\|x'\| \leqq 1$, and $i = 1, 2, \ldots$. Hence ([DS 5], 1.3Th.) the set $\{x'm_i, x' \in X', \|x'\| \leqq 1, i = 1, 2, \ldots\}$ is relatively weakly compact as a subset in $ca(B[a,b])$. We shall prove (cf. [DS 5], 2.9. Th.) that if

$$R = \{m_i(E), E \in B[a,b], i = 1, 2, \ldots\}$$

is regarded as a subset of X'' in the natural embedding, then R is relatively compact in the weak topology of X''. Since the embedding of X into X'' is closed in this topology, the statement will follow. From the preceding we obtain that the mappings $U_i : X' \to ca(B)$ defined by $U_i x' = x'm_i$ are equi-weakly compact operators, hence the adjoint operators $U_i' : ca'(B) \to X''$ are also equi-weakly compact. But the unit sphere of the dual space $ca'(B)$ contains the linear functionals $\{c_E : E \in B\}$, $c_E(\lambda) = \lambda(E)$, $\lambda \in ca(B)$. But $U_i'\{c_E : E \in B\} = \{m_i(E) : E \in B, i = 1, 2, \ldots\} \subset X''$, and R is therefore relatively weakly compact in X.

From the preceding lemma and Proposition the following lemma can be obtained.

Lemma 2. *Every sequence of vector functions of uniformly bounded variation is a sequence of vector functions of uniformly weakly compact semi-variation.*

Proof of Helly theorem bis. To prove Helly theorem bis it suffices to show that limit function of Helly theorem is of bounded variation. Let g be a weak limit of sequence g_n.Then for a fixed i, there exists x_i' in X' with $\|x_i'\| = 1$ such that [DS 5, II.3. 14 Cor.]

$$\|g(t_i) - g(t_{i-1})\| = |x_i'(g(t_i) - g(t_{i-1}))|$$

$$= \lim_n |x_i'(g_n(t_i) - g_n(t_{i-1}))|$$

Further

$$\sum_i \|g(t_i) - g(t_{i-1})\| = \sum_i |x_i'(g(t_i) - g(t_{i-1}))| =$$

$$\lim_n \sum_i |x_i'(g_n(t_i) - g_n(t_{i-1}))| \leqq$$

$$\limsup_n \sum_i \|g_n(t_i) - g_n(t_{i-1})\| \leqq K < \infty$$

for some positive K.

3. Moment problem theorem

Of all possible moment problems we shall consider only a power moment problem. We can formulate a task in the considered case as follows: Decide under which conditions there exists such a function of bounded variation $g(t) : [a,b] \to Y$ that

$$\int_a^b t^n dg(t) = y_n, \quad n = 0, 1, \ldots \tag{1}$$

We shall derive a concrete result relating to a power moment problem in the interval $[0, 1]$.

Theorem. *In order that there exists a function of bounded variation* $g(t) : [0,1] \to Y$ *such that*

$$\int_0^1 t^n dg(t) = y_n, y_n \in Y, n = 0, 1, \dots \tag{2}$$

it is necessary and sufficient that there exists a constant M *such that*

$$\sum_{k=0}^{n} \binom{n}{k} ||\Delta^{n-k} y_k|| \leq M, n = 0, 1, \dots, \tag{3}$$

where $\Delta^m y_k$ *denotes the* m*-th differences for the sequence* (y_k) *defined inductively by equalities*

$$\Delta^{m+1} y_k = \Delta^m y_k - \Delta^m y_{k+1}, \quad \Delta^0 y_k = y_k,$$

$$m = 0, 1, \dots; k = 0, 1, \dots. \tag{4}$$

Proof. Let the moment problem (2) be solved. Denote by L a bounded linear operator on the space $C[0,1]$ generated by a function of bounded variation, $g(t) : [0,1] \to Y$,

$$L(f) = \int_0^1 f(t) dg(t), \quad f \in C[0,1]$$

Put

$$x_k^{(m)}(t) = t^k (1-t)^m, \quad m, k = 0, 1, \dots. \tag{5}$$

Since

$$x_k^{(m+1)}(t) = t^k (1-t)^{m+1} = t^k (1-t)^m - t^{k+1} (1-t)^m =$$

$$x_k^{(m)}(t) - x_{k+1}^{(m)}(t),$$

we have

$$L(x_k^{(m+1)}) = L(x_k^{(m)}) - L(x_{k+1}^{(m)}), m, k = 0, 1, \dots.$$

Further

$$L(x_k^{(0)}) = y_k.$$

If we take into the consideration (4), we can easily see (by induction) that

$$L(x_k^{(m)}) = \Delta^m y_k, \quad m, k = 0, 1, \dots.$$

From this we deduce that (3) is satisfied. For we have

$$\sum_{k=0}^{n} \binom{n}{k} ||\Delta^{n-k} y_k|| =$$

$$\sum_{k=0}^{n} \binom{n}{k} || \int_0^1 x_k^{(n-k)}(t) dg(t) || \leq$$

$$\sum_{k=0}^{n}\binom{n}{k}\int_0^1 |x_k^{n-k)}(t)|d|g|(t) =$$

$$\int_0^1 (\sum_{k=0}^{n}\binom{n}{k}t^k(1-t)^{n-k})d|g|(t) =$$

$$= \int_0^1 [t+(1-t)]^n d|g|(t) \leq |g|([0,1]), \quad n = 1,2,\ldots$$

which proves the necessity of the condition, if we put $M = |g|([0,1])$.

The sufficiency. Let L_0 denote the operator defined on the set of functions (x_n), $x_n(t) = t^n, \quad n = 0,1,\ldots$ into Y by formula $L_0(x_n) = y_n, \quad n = 0,1,\ldots$. Extend L to the linear hull of the set (x_n), i. e. to the set of all polynomials. Namely if $x(t) = c_0 + c_1 t + \cdots + c_n t^n$, we put

$$L(x) = c_0 y_0 + c_1 y_1 = \cdots + c_n y_n.$$

Since the functions x_n are linearly independent the definition of L will be unique.

The operator L defined above will be clearly additive and homogeneous. We shall see that condition (3) implies that L is bounded.

Let us note that (even without this condition) the operator L is bounded on the set P_m of polynomials the degree of which is $\leq m$, because P_m is finite-dimensional space (as coordinates we take coefficients of polynomial), hence the convergence in P_m is coordinatewise.

We have

$$L(x_k^{(s)}) = \Delta y_k^s, \quad s,k = 0,1,\ldots.$$

Take any polynomial $p(t)$. Let the degree of $p(t)$ be m. Form the sequence of corresponding Bernstein polynomials (of $p(t)$)

$$p_n(t) = B_n(p;t) = \sum_{k=0}^{n}\binom{n}{k}p(\frac{k}{n})t^k(1-t)^{n-k}.$$

It is well-known that the degree of the polynomial $p_n(t)$ for any $n = 1,2,\ldots$ is not greater than m, and since $p_n(t)$ uniformly converges to $p(t)$ for $n \to \infty$, we have (according to remarks above) $L(p_n) \to L(p)$.

Let $(y_k, k \in N)$ satisfy the condition (3). For each positive integer n define a step function g_n with jumps at $\frac{m}{n}$ for $m = 0,1,\ldots,n-1$ by the following process. Let

$$y(j,n) := \binom{n}{j}\Delta^{n-j}y_j$$

for $j = 0,1,\ldots,n-1$. Set $g_n(0) = 0, g_n(1) = y_0$, and

$$g_n(x) := \sum_{j=0}^{m-1} y(j,n) \quad (\frac{m-1}{n} < x < \frac{m}{n}).$$

Extend g_n to [0,1] by averaging g_n at all jumps.

For each polynomial $P(x) = \sum_{j=0}^{n} c_j x^j$ put

$$\Lambda(P) = \sum_{j=0}^{n} c_j y_j.$$

Consider the Bernstein polynomials

$$B(k,n)(x) := \sum_{j=0}^{n} \binom{n}{j} (\frac{j}{n})^k x^k (1-x)^{n-j},$$

and observe that

$$\Lambda(B(k,n)) = \int_0^1 t^k dg_n(t) \tag{6}$$

for $n, k \in N$.

Since $(y_k, k \in N)$ satisfies the condition (3), it is clear that

$$\sum_{j=0}^{n} ||y(j,n)|| \leq L.$$

Hence the functions $g_0, g_1, \ldots$ are uniformly of bounded variation on $[0,1]$ with variation $\leq L$. Therefore by Helly theorem bis there is a function g of bounded variation such that $g_{n_i}(x) \to g(x), i \to \infty$ for x belonging to a dense subset of $[0,1]$. Then by Helly-Bray theorem [DDD 2]

$$\lim_{j\to\infty} \int_0^1 t^k dg_{n_j}(t) = \int_0^1 t^k dg(t)$$

Therefore by the formula (6) it suffices to show that

$$\lim_{n\to\infty} \Lambda(B(k,n)) = y_k$$

for $k \in N$. The proof is the same as in [DDD 2].

4. Integral representation theorem

It is well known that every continuous linear form on the space of $C(I)$ is presentable in the form

$$f \to \int_0^1 f(t) dq(t)$$

where q is a function of bounded variation.

From the preceding moment problem theorem we obtain the result giving representaion of majorated mapping [DI 3].

For each subset A of $[a,b]$, let $C([a,b], A)$ denote the space of continuous functions on $[a,b]$ vanishing outside A. If $F : C([a,b]) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum ||F(\psi_i)||$$

where supremum is over all finite families ψ_i in $C([a,b],A)$ with $\sum|\psi_i| \leq \chi_A(t)$ for all t in $[a,b]$.

If $F : C([a,b]) \to X$ is a linear mapping, then if

$$|||F_A||| < \infty$$

for all A in $\mathcal{B}([a,b])$ the mapping F is called majorated (also dominated, ([DI 3])).

An equivalent definition is as follows.

If $F : C([a,b]) \to X$ is a linear mapping, it is majorated (dominated) if and only if there exists a nonnegative Borel measure μ on $B[0,1]$ such that

$$||F(\psi)|| \leq \int_a^b |\psi(t)|d\mu(t),\ \psi \in C([a,b]).$$

We can obtain a representation theorem for majorated operators from the ([Di, 3]).

Theorem M. *Every majorated linear operator L on $C(I)$ into Y is presentable in the form*

$$L(f) = \int_0^1 f(t)dq(t) \tag{1}$$

where q is a function of bounded variation on I into Y. Conversely, every mapping of the form (1) *is majorated.*

Thus (1) represents majorated linear operators on $C(I)$ with values in Y.

Proof. Put

$$L(t^n) = y_n, \qquad n = 0,1,\ldots$$

Take

$$\sum_{k=0}^{n}\binom{n}{k}||\Delta^{n-k}y_k|| = \sum_{k=0}^{n}\binom{n}{k}||L(t_k(1-t)^{n-k})|| =$$

$$\sum_{k=0}^{n}||L(\binom{n}{k}t_k(1-t)^{n-k})|| \leq M < \infty$$

since L is majorated. Hence by preceding theorem

$$L(t^n) = \int_0^1 t^n dq(t)$$

for some function q of bounded variation on I with values in Y. By Weierstrass's theorem we can extend the last equality for every continuous function on I. $\square$

We have an equivalent form of the moment theorem (cf [DH 4], [H 6], [Wid 7]).

Theorem. *In order that there exists a vector measure of bounded variation $\boldsymbol{m}$: $B[0,1] \to Y$ such that*

$$\int_0^1 t^n d\boldsymbol{m}(t) = y_n, y_n \in Y, n = 0,1,\ldots$$

it is necessary and sufficient that there exists a constant $M < \infty$ such that

$$\sum_{k=0}^{n}\binom{n}{k}||\Delta^{n-k}y_k|| \leq M, n = 0,1,\ldots,$$

References

1. [BDS] BARTLE, R. G., DUNFORD, N., SCHWARTZ, J., *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.
2. [DDD] DEBIEVE, C., DUCHOŇ, M., DUHOUX, M., *A Helly theorem in the setting of Banach spaces*, Tatra Mount. Math. Publications **22** (2001), 105–114.
3. [DI] DINCULEANU, N., *Vector measures*, VEB, Berlin, 1966.
4. [DH] DUCHOŇ, M., HALUŠKA, J., *Moment problem for majorated operators*, Tatra Mount. Math. Publications 19 (2000), 177–182.
5. [DS] DUNFORD, N., SCHWARTZ, J., *Linear Operators, Part I: General Theory*, Interscience Publishers, Inc., New York 1958.
6. [H] HAUSDORFF, F., *Momentprobleme für ein endliches Interval*, Math. Z. **16** (1923), 220–248.
7. [W] WIDDER, D. V., *The Laplace transform*, Princeton University Press, Princeton, 1946.

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