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# ON THE GENERALIZED CONTINUITY OF THE SEMIVARIATION IN LOCALLY CONVEX SPACES

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If Condition (GB), introduced in [7] and [8] is fulfilled, then the everywhere convergence of the net of measurable functions implies the convergence of these functions with respect to the semivariation on a set of the finite variation of the measure $\mathbf{m}\colon \Sigma \to L(\mathbf{X}, \mathbf{Y})$, where $\Sigma$ is a $\sigma$-algebra of subsets of the set $T \neq \emptyset$, $\mathbf{X}, \mathbf{Y}$ are both locally convex spaces. The generalized strong continuity of the semivariation of the measure, introduced in this paper, implies Condition (GB).

## Introduction

The notion of the continuity of the semivariation of the measure is needed in many occasions in the integration theory with respect to the operator valued measure, cf. [4], e.g. convergence theorems are based on this notion in countable additive case of operator valued measure countable additive in the strong operator topology, cf. also [2] and [5]. Our notion of the generalized continuity of the semivariation of the measure enables us to develope a concept of an integral with respect to the $L(\mathbf{X}, \mathbf{Y})$-valued measure based on the net convergence of simple functions, where both $\mathbf{X}, \mathbf{Y}$ are locally convex topological spaces and $L(\mathbf{X}, \mathbf{Y})$ denotes the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$. Of course, the using of nets instead sequences leads to the generalization of the notion of the continuity of the measure which is sufficiently "fine". More precisely, we use the notion of the inner semivariation for this generalization. This way we restrict the set of $L(\mathbf{X}, \mathbf{Y})$-valued measures which can be taken for such type of integration. For instance, every atomic measure is generalized continuous. So, the class of measures with the generalized continuous semivariation is nonempty.

For terminology concerning the nets cf.[9]. For Condition (GB) cf. [7], [8].

If Condition (GB) is fulfilled, then the everywhere convergence of a net of measurable functions implies the convergence of these functions in semivariation on a set of finite variation. The generalized strong continuity of the semivariation implies Condition (GB). The classical Lebesque measure does not satisfy Condition (GB).

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## 1. Definitions

Let $T \neq \emptyset$ be a set and let $\Sigma$ be a $\sigma$-algebra of subsets of $T$. By $2^T$ we denote the potential set of $T$. Let $\mathbf{X}, \mathbf{Y}$ be two Hausdorff locally convex topological vector spaces. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms which define the topologies on $\mathbf{X}$ and $\mathbf{Y}$, respectively. $\mathbb{N} = \{1, 2, \dots\}$. Let $\mathbf{m}\colon \Sigma \to L(\mathbf{X}, \mathbf{Y})$ be an operator valued measure $\sigma$-additive in the strong operator topology (i.e. if $E \in \Sigma$, then $\mathbf{m}(E)\mathbf{x}$ is an $\mathbf{Y}$-valued measure for every $\mathbf{x} \in \mathbf{X}$).

**Definition 1.1.** *Let $p \in \mathcal{P}, q \in \mathcal{Q}$.*

(a) *By the $p,q$-semivariation of the measure* $\mathbf{m}$, cf. [1], *we mean a set function $\hat{\mathbf{m}}_{p,q}\colon \Sigma \to [0, \infty]$ defined as follows:*

$$\hat{\mathbf{m}}_{p,q}(E) = \sup q\left(\sum_{n=1}^{N} \mathbf{m}(E_n)\mathbf{x}_n\right), E \in \Sigma,$$

*where the supremum is taken over all finite disjoint partitions $\{E_n \in \Sigma; E = \bigcup_{n=1}^{N} E_n, E_n \cap E_m = \emptyset, n \neq m,\ n, m = 1, 2, \dots, N\}$ of $E$, and all finite sets $\{\mathbf{x}_n \in \mathbf{X}; p(\mathbf{x}_n) \leq 1, n = 1, 2, \dots, N\}, N \in \mathbb{N}$.*

(b) *By the $p,q$-variation of the measure $\mathbf{m}$ we mean a set function $\mathbf{v}_{p,q}(\mathbf{m}, .)\colon \Sigma \to [0, \infty]$, defined by the equality*

$$\mathbf{v}_{p,q}(\mathbf{m}, E) = \sup \sum_{n=1}^{N} q_p\left(\mathbf{m}(E_n)\right), E \in \Sigma,$$

*where the supremum is taken over all finite disjoint partitions $\{E_n \in \Sigma; E = \bigcup_{n=1}^{N} E_n, E_n \cap E_m = \emptyset, n \neq m,\ n, m = 1, 2, \dots, N,\ N \in \mathbb{N}\}$ of $E$ and*

$$q_p(\mathbf{m}(E)) = \sup_{p(\mathbf{x}) \leq 1} q(\mathbf{m}(E)\mathbf{x}).$$

(c) *By the inner $p,q$-semivariation of the measure $\mathbf{m}$ we mean a set function $\hat{\mathbf{m}}^*_{p,q} : 2^T \to [0, \infty]$, defined as follows: $\hat{\mathbf{m}}^*_{p,q}(F) = \sup_{E \subset F, E \in \Sigma} \hat{\mathbf{m}}_{p,q}(E), F \in 2^T$.*

**Lemma 1.2.** *The $p,q$-(semi)variation of the measure $\mathbf{m}$ is a monotone and $\sigma$-additive ($\sigma$-subadditive) set function, and $\mathbf{v}_{p,q}(\mathbf{m}, \emptyset) = 0, (\hat{\mathbf{m}}_{p,q}(\emptyset) = 0)$, for every $p \in \mathcal{P}$ and $q \in \mathcal{Q}$.*

**Proof.** Trivial.

**Definition 1.3.**

(a) *A set $E \in \Sigma$ is said to be of positive variation if there exist $q \in \mathcal{Q}, p \in \mathcal{P}$, such that $\mathbf{v}_{p,q}(\mathbf{m}, E) > 0$.*

(b) *We will say that $E \in \Sigma$ is a $\hat{\mathbf{m}}$-null set if $\hat{\mathbf{m}}_{p,q}(E) = 0$ for every $q \in \mathcal{Q}, p \in \mathcal{P}$.*

(c) *We say that a set $E \in \Sigma$ is of finite variation of the measure $\mathbf{m}$ if to every $q \in \mathcal{Q}$ there exists a $p \in \mathcal{P}$, such that $\mathbf{v}_{p,q}(\mathbf{m}, E) < \infty$. We will denote this relation shortly $\mathcal{Q} \to_E \mathcal{P}$, or, $q \mapsto_E p, q \in \mathcal{Q}, p \in \mathcal{P}$.*

**Remark 1.4.** The relation $\mathcal{Q} \to_E \mathcal{P}$ in Definition 1.3(c) may be different for different sets of finite variation of the measure $\mathbf{m}$.

**Definition 1.5.** *A measure* $\mathbf{m}$ *is said to satisfy Condition* (GB) *if for every* $E \in \Sigma$ *of finite and positive variation and every net of sets* $E_i \in \Sigma$, $E_i \subset E, i \in I$, *there holds*

$$\limsup_{i\in I} E_i \neq \emptyset$$

*whenever there exist real numbers* $\delta(q,p,E) > 0, p \in \mathcal{P}, q \in \mathcal{Q}$, *such that* $\hat{\mathbf{m}}_{p,q}(E_i) \geq \delta(q,p,E)$ *for every* $i \in I$ *and every couple* $(p,q) \in \mathcal{P} \times \mathcal{Q}$, *such that* $q \mapsto_E p$.

**Definition 1.6.** *We say that a set* $E \in \Sigma$ *of positive variation is an* $\hat{\mathbf{m}}$*-atom if every subset* $A$ *of* $E$ *is either* $\emptyset$ *or* $A \notin \Sigma$. *We say that a measure* $\mathbf{m}$ *is atomic if each* $E \in \Sigma$ *can be expressed in the form* $E = \bigcup_{n=1}^{\infty} A_n$, *where* $A_n, n \in \mathbb{N}$, *are* $\hat{\mathbf{m}}$*-atoms.*

## 2. On the generalized continuity of the semivariation

**Theorem 2.1.** *If* $\hat{\mathbf{m}}$ *is a continuous semivariation of an* $L(\mathbf{X},\mathbf{Y})$*-valued measure* $\mathbf{m}$, $\sigma$*-additive in the strong operator topology, where* $\mathbf{X},\mathbf{Y}$ *are both Banach spaces, then* $\mathbf{m}$ *satisfies Condition* (GB) *for sequences.*

**Proof.** Let $E \in \Sigma, 0 < \mathbf{v}(\mathbf{m},E) < \infty$ be given. Let $E_n \in \Sigma, E_n \subset E, n \in \mathbb{N}$, be a sequence of sets, such that there exists $\delta > 0$, such that $\hat{\mathbf{m}}(E_n) \geq \delta$ for every $n \in \mathbb{N}$. We have:

$$\limsup_{n\in\mathbb{N}} E_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m = \bigcap_{n=1}^{\infty} H_n,$$

where $H_n = \bigcup_{m=n}^{\infty} E_m \in \Sigma, n \in \mathbb{N}$, is a nonincreasing set sequence. Since $\hat{\mathbf{m}}$ is continuous, there is:

$$\hat{\mathbf{m}}(\limsup_{n\in\mathbb{N}} E_n) = \hat{\mathbf{m}}(\bigcap_{n=1}^{\infty} H_n) = \lim_{n\in\mathbb{N}} \hat{\mathbf{m}}(H_n). \tag{1}$$

Since $\hat{\mathbf{m}}(E_n) \geq \delta$ and $E_n \subset H_n$ for every $n \in \mathbb{N}$, there is $\hat{\mathbf{m}}(H_n) \geq \delta$ for every $n \in \mathbb{N}$. Thus (1) implies $\hat{\mathbf{m}}(\limsup_{n\in\mathbb{N}} E_n) \geq \delta$, i.e. $\limsup_{n\in\mathbb{N}} E_n \neq \emptyset$.

**Lemma 2.2.** *Let* $\Sigma$ *contains all singletons (= one-point subsets) of a set* $E \in \Sigma$ *of positive and finite variation and* $\hat{\mathbf{m}}_{p,q}(\{t\}) = 0$ *for every* $t \in E$ *and every couple* $(p,q) \in \mathcal{P} \times \mathcal{Q}$, *such that* $q \mapsto_E p$. *Then Condition* (GB) *is not satisfied.*

**Proof.** Denote by $E_{t_1,\ldots,t_n} = E \setminus \{t_1,\ldots,t_n\}, n \in \mathbb{N}$. Then the net $(E_{t_1,\ldots,t_n}; t_1,\ldots, t_n \in E, n \in \mathbb{N})$ is a nondecreasing net of subsets of the set $E_i$ with $\bigcap_{t_1,\ldots,t_n\in E} E_{t_1,\ldots,t_n} = \emptyset$. Show that

$$\hat{\mathbf{m}}_{p,q}(E) = \hat{\mathbf{m}}_{p,q}(E_{t_1,\ldots,t_n}) \tag{2}$$

for every $p \in \mathcal{P}, q \in \mathcal{Q}$, and every $t_i, \ldots t_n \in E, n \in \mathbb{N}$.

We have:

$$\begin{aligned}\hat{\mathbf{m}}_{p,q}(E) &= \hat{\mathbf{m}}_{p,q}([E \setminus \{t_1,\dots,t_n\}] \cup \{t_1,\dots t_n\}) \\ &\le \hat{\mathbf{m}}_{p,q}(E \setminus \{t_1,\dots,t_n\}) + \sum_{i=1}^{n} \hat{\mathbf{m}}_{p,q}(\{t_i\}) \\ &= \hat{\mathbf{m}}_{p,q}(E \setminus \{t_1,\dots,t_n\}),\end{aligned}$$

for every $q \in \mathcal{Q}, p \in \mathcal{P}$, and every finite set $\{t_1,\dots,t_n\}, n \in \mathbb{N}$, of points from $E$. The inverse inequality follows from the fact that $\hat{\mathbf{m}}_{p,q}, p \in \mathcal{P}, q \in \mathcal{Q}$, is a monotone set function.

We found the net $(E_{t_1,\dots,t_n} \in \Sigma, t_1,\dots,t_n \in E, n \in \mathbb{N})$ of sets with the empty intersection and such that $\hat{\mathbf{m}}_{p,q}(E_{t_1,\dots,t_n}) = \hat{\mathbf{m}}_{p,q}(E) = \delta(q,p,E) > 0$ for some $p \in \mathcal{P}, q \in \mathcal{Q}$, such that $q \mapsto_E p$. So, Condition (GB) is not satisfied.

**Corollary 2.3.** *The Lebesgue measure does not satisfy Condition* (GB) *on the real line (for arbitrary nets of sets).*

**Corollary 2.4.** *If an operator valued measure* $\mathbf{m}$ *satisfies Condition* (GB)*, then for every* $E \in \Sigma$ *of positive and finite variation there exists a singleton* $\{t\}, t \in E$*, such that* $\hat{\mathbf{m}}_{p,q}(\{t\}) > 0$ *for some* $p \in \mathcal{P}, q \in \mathcal{Q}$*. Such singleton is clearly an* $\hat{\mathbf{m}}$*-atom.*

**Definition 2.6.** *We say that the semivariation of the measure* $\mathbf{m} : \Sigma \to L(\mathbf{X},\mathbf{Y})$ *is* GS- *(= generalized strongly) continuous if for every set of finite variation* $E \in \Sigma$ *and every monotone net of sets* $E_i \subset T, E_i \subset E, i \in I$*, the following equality*

$$\lim_{i\in I} \hat{\mathbf{m}}^*_{p,q}(E_i) = \hat{\mathbf{m}}^*_{p,q}(\lim_{i\in I} E_i)$$

*holds for every couple* $(p,q) \in \mathcal{P} \times \mathcal{Q}$*, such that* $q \mapsto_E p$*.*

**Theorem 2.7.** *If the semivariation of a measure* $\mathbf{m}\colon \Sigma \to L(\mathbf{X},\mathbf{Y})$ *is* GS*-continuous, then the measure* $\mathbf{m}$ *satisfies Condition* (GB)*.*

**Proof.** Let $E \in \Sigma$ be a set of positive and finite variation. Let $E_i \subset E$, $\hat{\mathbf{m}}^*_{p,q}(E_i) \ge \delta = \delta(q,p,E) > 0$ for every $i \in I$. We have:

$$\limsup_{i\in I} E_i = \bigcap_{i\in I} \bigcup_{j\ge i} E_j = \bigcap_{i\in I} H_i,$$

where $H_i, i \in I$, is a nonincreasing net of sets. So, by assumption

$$\begin{aligned}\lim_{i\in I} \hat{\mathbf{m}}^*_{p,q}(H_i) &= \hat{\mathbf{m}}^*_{p,q}(\lim_{i\in I} H_i) \\ &= \hat{\mathbf{m}}^*_{p,q}(\bigcap_{i\in I} H_i) \\ &= \hat{\mathbf{m}}^*_{p,q}(\limsup_{i\in I} E_i). \qquad (3)\end{aligned}$$

Since $\hat{\mathbf{m}}^*_{p,q}(E_i) \geq \delta$ and $H_i \supset E_i$ for every $i \in I$, we have $\hat{\mathbf{m}}^*_{p,q}(H_i) \geq \delta$, too. Then (3) implies $\hat{\mathbf{m}}^*_{p,q}(\limsup_{i\in I} E_i) \geq \delta$. Recall the definition of the inner $p,q$-semivariation:

$$\hat{\mathbf{m}}^*_{p,q}(F) = \sup_{E\subset F, E\in\Sigma} \hat{\mathbf{m}}_{p,q}(E), F \in 2^T.$$

So, there is a couple $(p,q) \in \mathcal{P} \times \mathcal{Q}, q \mapsto_E p$, such that for every $\varepsilon > 0, \delta > \varepsilon > 0$, there exists a set $G \in \Sigma, G \subset \limsup_{i\in I} E_i$, such that

$$0 < \delta - \varepsilon \leq \hat{\mathbf{m}}^*_{p,q}(\limsup_{i\in I} E_i) - \varepsilon \leq \hat{\mathbf{m}}_{p,q}(G),$$

and, therefore, $G \neq \emptyset$, i.e. $\limsup_{i\in I} E_i \neq \emptyset$ and Condition (GB) is satisfied.

**Example 2.8.** Let $\mathbf{m} : \Sigma \to L(\mathbf{X}, \mathbf{Y})$ be an (countable) atomic measure. Show that the semivariation of the measure $\mathbf{m}$ is GS–continuous. Indeed, let $E \in \Sigma$ be a set of finite and positive variation. Let $E_i, i \in I$, be an arbitrary decreasing net of sets. Recall that $E_i \searrow G(\in 2^G) \Rightarrow (1) i < j \Rightarrow E_i \supset E_j, (2) \bigcap_{i\in I} E_i = G$. It is claear that it is enought to consider the case $G = \emptyset$, because $E_i \searrow G \Leftrightarrow G \subset E_i, E_i \setminus G \searrow \emptyset$.

In the case $E_i \in \Sigma, i \in I$, there is

$$\lim_{i\in I} \hat{\mathbf{m}}^*_{p,q}(E_i) = \lim_{i\in I} \hat{\mathbf{m}}_{p,q}(E_i), \tag{4}$$

and since the family of atoms is at most a countable set, there is $\lim_{i\in I} E_i = \bigcap_{i\in I} E_i \in \Sigma$ and, therefore,

$$\hat{\mathbf{m}}^*_{p,q}(\lim_{i\in I} E_i) = \hat{\mathbf{m}}_{p,q}(\lim_{i\in I} E_i) \tag{5}$$

for every $p \in \mathcal{P}, q \in \mathcal{Q}$.

Take an arbitrary set $E \in \Sigma$ of the positive and finite variation. Denote by $\mathcal{A}$ the set of all $\hat{\mathbf{m}}$-atoms. Denote by $l(i,E) = (\mathcal{A} \cap E) \setminus E_i, i \in I$. Clearly $i < j, i, j \in I \Rightarrow l(i,E) \leq l(j,E)$ and there exist atoms $A_n \in \mathbb{N}$, such that $l(i,E) = \{A_1, A_2, \ldots, A_n, \ldots\}$.

Let $\varepsilon > 0$ be given. Since $\hat{\mathbf{m}}^*_{p,q}(E) = \text{const}(p,q) < \infty, q \mapsto_E p$, there is a set $E_i, i \in I$, such that

$$\hat{\mathbf{m}}^*_{p,q}(E_i) = \text{const}(p,q) - \sum_{A_n \in l(i,E)} \hat{\mathbf{m}}^*_{p,q}(A_n) < \varepsilon, q \mapsto_E p, p \in \mathcal{P}, q \in \mathcal{Q}. \tag{6}$$

And the inequality (6) holds for every $E_j \supset E_i, i, j \in I$. Combining (4), (5), (6) and Definition 2.6. we see that the assertion is proved for the case when $E_i \in \Sigma, i \in I$, is a decreasing net of sets. The case of the increasing net of sets can be proved analogously.

Let now $G \subset T$ be an arbitrary set. So, there is exact one (countable) set $E^* = \mathcal{A} \cap G$ with the property:

$$\hat{\mathbf{m}}^*_{p,q}(G) = \sup_{E\subset G, E\in\Sigma} \hat{\mathbf{m}}^*_{p,q}(E) = \hat{\mathbf{m}}_{p,q}(E^*), p \in \mathcal{P}, q \in \mathcal{Q}.$$

The proof for the inner measure and the arbitrary net of subsets $E_i, i \in I$, we obtain now repeating the previous procedure of the proof concerning the measure $\mathbf{m}$ and the set system $\Sigma$.

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