Preface

Intervals in music are rather to be judged intellectually through numbers than sensibly through the ear.

Pythagoras

Definition of tone system The sculptor working in marble has set his limit by the choice of this material to the exclusion of all other materials. Analogously, the musician has to select the tone system he wants to use. This strong emphasis on the necessity of limitation reflects not a subjective prejudice but it is a fundamental artistic law. There is no art without limitation.

There are four important and mutually interacting attributes that we can manipulate to create or describe any sound. And we can work with these attributes in two different ways: we can measure them and we can hear them. If we measure them, they are physical attributes; if we hear them, they are perceptual attributes. The four physical attributes are: frequency, amplitude, waveform, and duration. Their perceptual counterparts are: pitch, loudness, timbre, and (psychological) time. There is similarity between hearing and measuring these attributes; however, it is a complex correlation. The two are not exactly parallel.

There are more or less suitable mathematical structures that reflect the nature of the various classes $\mathcal{T}$ of tones and distinguish them from other hearable sound. Dealing with tones as physical sound, elements of $\mathcal{T}$ can be modelled as Fourier series (Chapter 2) or wavelets (Chapter 3). Considering tones as perceptual objects, classes $\mathcal{T}$ are usually mentioned as the sets of numbers, e.g., the set of all integer numbers $\mathbb{Z}$ (Equal Temperaments, c.f. Chapter 5), the set of all fuzzy numbers $\mathbb{F}$ (Well Temperaments, c.f. Chapter 7), the set of all numbers of the form 2^p3^q, where $p, q \in \mathbb{Z}$ (Pythagorean System, c.f. Chapter 6) or $p, q \in \mathbb{Q}$ (Euler music space, c.f. Chapter 4), or, they are mentioned as a finite sets which are "acoustically dense" (53-tones per octave in Turkish music, c.f. Chapter 5).

What is tone system from the mathematical viewpoint? Having defined the *class of tones* $\mathcal{T}$ (what tones are), we can start with the following very simplified definition.

Definition 1 Let $\mathcal{T}$ be a class of tones. *Tone system in a broader sense* is a couple $(\mathbb{T}, \Omega)$ where $\mathbb{T}$ is a subset of $\mathcal{T}$ and $\Omega : \mathcal{T} \to \mathbb{R}$ is a real function called the *pitch function*. The set $S = \Omega(\mathbb{T}) = \{\Omega(T) \in \mathbb{R};\ T \in \mathbb{T}\}$ is called the *tone system in a narrower sense*.

There are many examples in this book showing the merits of Definition 1. However, it is clear that tone system in Definition 1 is such an abstract notion, so that we are not able to distinguish exactly which structures in nature and art are music-inspired or related to music and which are not. After some years of considerations of systems of tones used in music, our present opinion is that the "usual" tone system should involve the following four additional concepts which, in general, we formulate only verbally:

(S1) construction or selection algorithm for $\mathbb{T} \subset \mathcal{T}$;

(S2) notion of symmetry;

(S3) characteristic or typical relation, fundamental equation;

(S4) uncertainty measure.

On the one hand, Example 4, c.f. Chapter 1, is a collection of number sets such that no concepts (S1)–(S2) but trivial was found in these African tone systems. On the other hand, if the concepts (S1)–(S4) are nontrivial, then they are mathematically very heterogeneous and also qualitatively very different for different tone systems. An exemplary explicit formulation of (S1)–(S4) can be found in Chapter 7, Definition 24.

There are three main mutually interacting sources that lead to useful tone systems in music (in this book we do not consider applications of the tone system notion to other disciplines of science and art): *acoustics* (e.g., physical properties of sound, the building acoustics, spectral analysis, timbre, turbulence of air); *psychology* (e.g., the Weber–Fechner law, linear part of the pitch perception curve, dissonance curves, reference tone, individual perception); and *human spiritual culture* (e.g., language, art—including music and dance, philosophy, science, instrument building).

Unifying the theory The set $\mathcal{T}$ may be equipped with more unary (e.g., scalar multiplication), binary (e.g., addition, join, meet), ternary, etc., n-ary operations (e.g., this aggregation operation is produced by an n-member orchestra), and also with different orders (e.g., the linear order according to the frequency of tones, the "spiral of fifths", c.f. also List of intervals), topological, and other mathematical structures.

The sense of Definition 1 consists of the idea that the set $\Omega(\mathbb{T})$ of numbers (quantity) should reflect extracted mathematical structures of the set $\mathbb{T}$ of tones (quality). The pitch function Ω is an element of the "first dual" $\mathcal{T}'$ of $\mathcal{T}$ (should be defined exactly; to have a vector structure, tone pitches can be mentioned as logarithms of relative frequencies of tones). If the first dual is bounded with hearing, then the second dual $\mathcal{T}''$ is a manifestation of measuring. Or vice-versa, c.f. the psychological model in Chapter 4, Figure 4.1.

While the first extremal approach to create a unifying tone system theory (*the concentration on mere quantitative elements*—psychological statistics, physics of sound) isolates us from the inner world of music (and therefore, it is not unifying), the second approach (*the concentration on mere qualitative elements*—spiritual, musical, religious, metaphysical, philosophical) affects on few concrete tone systems that may be used practically. This second approach yields a number of particular, often contentious, theories about tone systems which are concentrated on one side of the topic: national music, production of musical instruments, philosophy, one music style.

The truth should be somewhere in between. This book is an attempt to describe elements of a unifying mathematical theory of tone systems on the basis of the *uncertainty-knowledge-based information theory.* On the other hand, the study of tone systems provides an excellent mathematical laboratory to study all types of uncertainty.

Moreover, the tone system notion mediates not only a special mathematical duality. Mathematics is very apt to reflect the *spiritual duality "natural science* $\leftrightarrow$ *art"* which has its manifestation in the notion of tone system, [39]. This duality was one of three main motivations for writing this book. Maybe the tone system notion is unique in this way. This duality has two directions ("natural science $\rightarrow$ art" and "art $\rightarrow$ natural science") which are mutually symbiotic and reflective.

The second motivation was finding out that the *interdisciplinarity is the inner property of tone systems* and that the topic of tone systems is a very mathematical subject. There are three general arguments for the unambiguous and resolute quoted claim of Pythagoras that we should care much more for tone systems: Theory of tone systems is a *very interdisciplinary topic* within mathematics itself and is only inspired by and applied to music. There are papers concerning the tone systems in set theory, harmonic analysis, number theory, fuzzy theory, genetic algorithms, statistics, graph theory, algebra, theoretic arithmetic, differential equations, Diophantine equations, dynamical systems, geometry, logic, discrete mathematics, functional analysis. The theory of tone systems *can be applied not only to music* but also to medicine, acoustics, linguistics, quantum mechanics, information science, and psychology. Problems about tone systems were studied at the *origin of mathematics as a science.* They are their own legacy of mathematics. Many results and even whole branches of mathematics (e.g., partial differential equations, Fourier analysis) arose from problems about tone systems during its history.

The third main motivation for writing this book was my opinion that *the most valuable technical advance for music in general is the development of tone systems.*

Part I: Fundamentals We explain that the essential approach for the study of tone systems is use of uncertainty-based information theory which gives us an integrated view on the subject and, at the same time, forms the theory of tone systems as an open system. From the contents it is immediately clear that there are connections with harmonic analysis, mathematical psychology, and fuzzy set and systems theory.

The notion of the *geometric net* (of numbers, operators) is a generalization of the elementary notion of geometric progression to more quotients. It was developed for describing tone systems. This technique is systematically used through the special systems part of the book. But we do not study systematically the theory of geometric net. The concept of geometric net is related to the notion of *analytic algebra* which is a generalization of power series.

The mathematical terminology is standard. It is worth noting two items. First, in the multiplicative group $(0, \infty)$, the length of an interval $[a, b]$, where $a \leq b$, is b/a. Borrowing the usual musical terminology, we will simply say that b/a is an interval. The second terminological peculiarity is the use of special historical names of some lengths of intervals or numbers, c.f. List of intervals. For the compilation of this list, various sources were used such as [26]. Intervals listed here were ordered by M. Op de Coul. For further names of intervals (also outside octave), c.f. e.g. [26] or the Bohlen–Pierce site

`http://members.aol.com/bpsite/index.html.`

A consistent system of nomenclature is described in

`http://uq.net.au/~ zzdkeena/Music/IntervalNaming.htm`

(D. Keenan: A note on the naming of musical intervals).

Part II: Special systems The rest of the chapters describes classes of tone systems. It should be underlined that we classified (or covered) all *practical* and also theoretical tone systems (in the narrower sense at least). The families of tone systems described in this part are not disjoint. On the contrary, there are many set and also idea intersections among the tone systems families.

At the end of the chapters there are “Remarks and ideas for exploring”. Here are noted also some suggestions to prove or calculate some additional or complementary facts to the basic text. Some of the exercises are unsolved problems and are challenges to make new discoveries.

There is a question of unambiguity and systemizing of tone systems in the literature. The names of tone systems in this book are built according to the following pattern:

NAME1/…/*NAMEp*(*COM1*/…/*COMq*)-N,
$\omega_1, \omega_2, \ldots, \omega_N$
where *NAMEp* are tone system names in the literature (often the creators' names); *COMq* are comment acronyms (not necessarily present), c.f. List of abbreviations and acronyms, making the scale unambiguously identified; N denotes the cardinality of the discrete tone system or the cardinality of its period if the tone system is periodical. If N is not discrete (e.g., the case of gamelan tone systems), then it is the number of continuous zones or continuous zones within a period. After the tone system name, there is a short text comment and then the sequence $\omega_1, \omega_2, \ldots, \omega_N$ of pitch values (or zone means) follows in the footnote size. The pitch values are either in cents or expressed as ratios (or both, mixed); they are not in Hz. If S is not periodical, then $S = \{\omega_1, \omega_2, \ldots, \omega_N\}$ and $\omega_1 = 1$ (0 cents). *The absence of value 1 (the unison) means that S is periodical.* In this case, the last file value is this period interval (mostly 2 =1 200 cents, the octave) with the exception that for tetrachordal scales, two periods are printed.

We illuminated the text with more than 200 tone systems used in practice (their names are alphabetically listed in the Index) and more than 800 systems are listed in Appendix A. An excellent collection of examples is a permanently updated archive of used tone systems created by the M. Op de Coul (together with other composers), the Fokker Foundation, c.f. [192], which contains more than 3000 items.

Literature The key figures from the past are: Archytas, Avicena, Euler, Fibonacci, Fourier, Helmholtz, Kepler, Mersenne, Petzval, Pólya, Ptolemy, Pythagoras, and Stevin. We collected *literature about tone systems reviewed in Mathematical Reviews* since its beginning in 1940. Most of these publications are listed in the bibliography. We omitted the popularized works and works about the history of tone systems, c.f. [193].

Although there are some short passages which aim to illuminate the correlation between time and the conceptual development of tone systems, the *book is not about the history of tone systems.* Rather, the most complete present special and permanently updated literature list about tone systems and their history is collected by M.

Op de Coul, B. McLaren, F. Jedrzejewski, [192]. This bibliography contains more than 4 000 items and also internet links to specialized journals and other internet pages.

There are many book sources concerning the *musicological terminology used in tone systems*. This book is self-contained. For those who are curious, a musicological dictionary with definitions that lead easily to mathematically formalized definitions of terms was created and is continually improved on the Web by the composer J. L. Monzo, c.f. [195].

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I hope that the book will be interesting and useful for mathematicians, musicians, teachers, tuners, producers of musical instruments, and university students. The book is specially recommended to those mathematicians who play some musical instrument.

Ján Haluška

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